

Not-for-Publication Appendix to: Forecast Rationality Tests in the Presence of Instabilities, With Applications to Federal Reserve and Survey Forecasts

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This Appendix consists of four sections. Section 1 lists the theoretical results listed in the paper, Section 2 provides their proofs. Section 3 provides additional critical values for the proposed tests and Section 4 plots the complete data used in the empirical application.

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1 List of Results To Be Proved

Mean Value Approximation. *The following mean value approximation holds:*

$$m^{1/2}\widehat{\theta}_j = G^{-1} \left(\frac{T}{m} \right)^{1/2} [I_\ell, FB] \left\{ \frac{1}{\sqrt{T}} \sum_{t=R}^j \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \frac{1}{\sqrt{T}} \sum_{t=R}^{j-m} \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} \right\} + A_j, \quad (1)$$

where $\lim_{T \rightarrow \infty} \sup_j A_j = o_p(1)$.

Proposition 1 (Preliminary Asymptotic Result) *Under Assumptions 1-4 and $T^{-1/2}\xi_T \rightarrow \xi$ in $\mathcal{D}_{\mathbb{R}(\ell+q)}[0, 1]$:*

$$\frac{1}{\sqrt{T}} \sum_{t=1}^j \begin{pmatrix} b_{R,t,j} \cdot I_\ell & 0 \\ 0 & a_{R,t,j} \cdot I_q \end{pmatrix} \begin{pmatrix} f_{t+h} \\ h_t \end{pmatrix} \Rightarrow \int_0^\tau \Omega(s, \tau)^{1/2} d\xi(s),$$

where $\xi(s) = S^{1/2}\mathcal{B}_{\ell+q}(s)$, $\mathcal{B}_{\ell+q}(s)$ is an $(\ell + q) \times 1$ vector of independent standard Brownian motions, \mathcal{D} denotes the space of cadlag functions, “ \Rightarrow ” denotes weak convergence with respect to the Skorohod metric, and $\Omega(s, \tau)^{1/2} \equiv \begin{pmatrix} \sigma_f(s) \cdot I_\ell & 0 \\ 0 & \sigma_h(s, \tau) \cdot I_q \end{pmatrix}$.

Proposition 2 (Asymptotic Distribution of $\widehat{\theta}_j$) *Under Assumptions 1-4 and $T^{-1/2}\xi_T \rightarrow \xi$ in $\mathcal{D}_{\mathbb{R}(\ell+q)}[0, 1]$:*

$$m^{1/2} \widehat{\theta}_j \Rightarrow \int_0^\tau \widetilde{\omega}(s, \tau) d\mathcal{B}_{\ell+q}(s) - \int_0^{\tau-\mu} \widetilde{\omega}(s, \tau - \mu) d\mathcal{B}_{\ell+q}(s) = \mathcal{B}_{\widetilde{\omega}}(\tau) - \mathcal{B}_{\widetilde{\omega}}(\tau - \mu) \quad (2)$$

$$\stackrel{d}{=} \int_0^\tau \omega(s, \tau) d\mathcal{B}_{\ell+q}(s) \equiv \mathcal{B}_\omega(\tau) = \mathcal{B}_\ell \left(\int_0^\tau \omega(s, \tau) \omega(s, \tau)' ds \right), \quad (3)$$

where

$$\widetilde{\omega}(s, \tau) = \mu^{-\frac{1}{2}} G^{-1} [I_\ell, FB] \Omega(s, \tau)^{1/2} S^{1/2}, \quad (4)$$

$$\begin{aligned} \omega(s, \tau) &= \mu^{-\frac{1}{2}} G^{-1} [I_\ell, FB] \left\{ \left[\Omega(s, \tau)^{1/2} - \Omega(s, \tau - \mu)^{1/2} \right] \cdot \mathbf{1}(s \leq \tau - \mu) \right. \\ &\quad \left. + \Omega(s, \tau)^{1/2} \cdot \mathbf{1}(\tau - \mu < s \leq \tau) \right\} S^{1/2}, \end{aligned} \quad (5)$$

$\mathcal{B}_{\ell+q}(s)$ is an $(\ell + q) \times 1$ vector of independent standard Brownian motions and $\stackrel{d}{=}$ denotes equality in distribution.

Proposition 3 (Calculation of $\int_0^\tau \omega(s, \tau) \omega(s, \tau)' ds$)

$$\begin{aligned} \int_0^\tau \omega(s, \tau) \omega(s, \tau)' ds &= \mu^{-1} G^{-1} \left\{ \left(\int_{\tau-\mu}^\tau \sigma_f^2(s) ds \right) S_{ff} + \left(\int_{\tau-\mu}^\tau \sigma_f(s) \sigma_h(s, \tau) ds \right) \right. \\ &\quad \left. (FBS_{fh} + S_{fh} B' F') + \int_0^\tau [(\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 \cdot 1(s \leq \tau - \mu) \right. \\ &\quad \left. + \sigma_h^2(s, \tau) \cdot 1(\tau - \mu \leq s \leq \tau)] ds FBS_{hh} B' F' \right\} G^{-1}, \end{aligned}$$

where

(i) $\int_{\tau-\mu}^\tau \sigma_f^2(s) ds = \mu$ for both rolling and recursive cases;

(ii) recursive: let $\tilde{\pi} \equiv \mu / (\tau - \mu)$;

$\int_{\tau-\mu}^\tau \sigma_f(s) \sigma_h(s, \tau) ds = \mu [1 - \tilde{\pi}^{-1} \ln(1 + \tilde{\pi})]$ and

$\int_0^\tau [(\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 \cdot 1(s \leq \tau - \mu) + \sigma_h^2(s, \tau) \cdot 1(\tau - \mu \leq s < \tau)] ds = 2\mu [1 - \tilde{\pi}^{-1} \ln(1 + \tilde{\pi})]$;

(iii) rolling: let $\pi^\dagger \equiv \frac{\mu}{\rho}$;

(a) if $\mu \geq \rho$, then

$\int_{\tau-\mu}^\tau \sigma_f(s) \sigma_h(s, \tau) ds = \mu \left(1 - \frac{1}{2\pi^\dagger}\right)$ and

$\int_0^\tau [(\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 \cdot 1(s \leq \tau - \mu) + \sigma_h^2(s, \tau) \cdot 1(\tau - \mu \leq s < \tau)] ds = \mu \left(1 - \frac{1}{3\pi^\dagger}\right)$;

(b) if $\mu < \rho$, then

$\int_{\tau-\mu}^\tau \sigma_f(s) \sigma_h(s, \tau) ds = \frac{1}{2} \mu \pi^\dagger$ and

$\int_0^\tau [(\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 \cdot 1(s \leq \tau - \mu) + \sigma_h^2(s, \tau) \cdot 1(\tau - \mu \leq s < \tau)] ds = \mu \pi^\dagger \left(1 - \frac{1}{3} \pi^\dagger\right)$.

Proposition 4 (Calculation of $\int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds$)

$$\begin{aligned} \int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds &= \mu^{-1} G^{-1} \left\{ \left(\int_0^\tau \sigma_f^2(s) ds \right) S_{ff} + \left(\int_0^\tau \sigma_f(s) \sigma_h(s, \tau) ds \right) \times \right. \\ &\quad \left. \times (FBS_{fh} + S_{fh} B' F') + \left(\int_0^\tau \sigma_h^2(s, \tau) ds \right) FBS_{hh} B' F' \right\} G^{-1}, \end{aligned}$$

where

(i) $\int_0^\tau \sigma_f^2(s) ds = (\tau - \rho)$ for both rolling and recursive cases;

(ii) recursive:

$\int_{\tau-\mu}^\tau \sigma_h(s, \tau) \sigma_f(s) ds = (\tau - \rho) \left(1 - \frac{\rho}{\tau - \rho} \ln\left(\frac{\tau}{\rho}\right)\right)$ and

$\int_0^\tau \sigma_h^2(s, \tau) ds = 2(\tau - \rho) \left(1 - \frac{\rho}{\tau - \rho} \ln\left(\frac{\tau}{\rho}\right)\right)$;

(iii) rolling:

(a) if $\tau - \rho \geq \rho$, then

$\int_0^\tau \sigma_f(s) \sigma_h(s, \tau) ds = (\tau - \frac{3}{2}\rho)$ and $\int_0^\tau \sigma_h^2(s) ds = (\tau - \frac{4}{3}\rho)$;

(b) if $\tau - \rho < \rho$, then

$\int_0^\tau \sigma_f(s) \sigma_h(s, \tau) ds = \frac{1}{2\rho} (\tau - \rho)^2$ and $\int_0^\tau \sigma_h^2(s) ds = \frac{1}{3\rho^2} (\tau - \rho)^2 (4\rho - \tau)$.

Theorem 5 (Main Proposition) Under Assumption 1-4,

$$\begin{aligned} \mathcal{W}_{j,m} &= m\widehat{\theta}_j' V_{\theta,\tau}^{-1}\widehat{\theta}_j \\ &\Rightarrow \left[\mathcal{B}_\ell \left(\int_0^\tau \omega(s,\tau) \omega(s,\tau)' ds \right) \right]' V_{\theta,\tau}^{-1} \left[\mathcal{B}_\ell \left(\int_0^\tau \omega(s,\tau) \omega(s,\tau)' ds \right) \right], \end{aligned} \quad (6)$$

where

$$V_{\theta,\tau} = Avar \left(m^{1/2}\widehat{\theta}_j \right) \quad (7)$$

$$\begin{aligned} &= G^{-1} [I_\ell, FB] Avar \left(\frac{1}{\sqrt{m}} \sum_{t=j-m+1}^j \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} \right) [I_\ell, FB]' G^{-1} \\ &= \int_0^\tau \omega(s,\tau) \omega(s,\tau)' ds, \end{aligned} \quad (8)$$

$j = [\tau T]$, $m = [\mu T]$ and $\mathcal{B}_\ell(\cdot)$ is a standard ℓ -dimensional Brownian motion. Let θ_j be the true parameter value. We reject the null hypothesis:

$$H_0 : \theta_j = \theta_0, \theta_0 = 0 \text{ for all } j = R + m, \dots, T \quad (9)$$

if $\max_{j \in \{R+m, \dots, T\}} \mathcal{W}_{j,m} > \kappa_{\alpha,\ell}$, where $\kappa_{\alpha,\ell}$ are the critical values at the $100\alpha\%$ significance level that can be simulated for given values of μ , ℓ , G , F , B and S .

Proposition 6 (Special Case I: Irrelevant Parameter Estimation Error) Under the condition $F = 0$, parameter estimation error becomes irrelevant and $\int_0^\tau \omega(s,\tau) \omega(s,\tau)' ds$ becomes $G^{-1} S_{ff} G^{-1}$ and $\int_0^\tau \tilde{\omega}(s,\tau) \tilde{\omega}(s,\tau)' ds = \frac{(\tau-\rho)}{\mu} G^{-1} S_{ff} G^{-1}$ for all estimation schemes.

Proposition 7 (Special Case II: Forecast Unbiasedness and Efficiency Tests) Under the condition:

$$S_{ff} = -\frac{1}{2}(FBS_{hf} + S_{fh}B'F') = FBS_{hh}B'F', \quad (10)$$

$\int_0^\tau \tilde{\omega}(s,\tau) \tilde{\omega}(s,\tau)' ds$ in Proposition 4 becomes:

(i) recursive case: $\frac{\tau-\rho}{\mu} G^{-1} S_{ff} G^{-1}$;

(ii) rolling case:

(a) $\frac{1}{\mu} \frac{2\rho}{3} G^{-1} S_{ff} G^{-1}$, if $\tau - \rho \geq \rho$; and

(b) $\frac{(\tau-\rho)}{\mu} \left(1 - \frac{(\tau-\rho)^2}{3\rho^2} \right) G^{-1} S_{ff} G^{-1}$, if $\tau - \rho < \rho$.

Furthermore, $\int_0^\tau \omega(s,\tau) \omega(s,\tau)' ds$ in Proposition 3 becomes $\lambda G^{-1} S_{ff} G^{-1}$, where:

(i') recursive case: $\lambda = 1$;

(ii') rolling case: let $\pi^\dagger \equiv \frac{\mu}{\rho}$; then,

(a) $\lambda = \frac{2}{3\pi^\dagger}$, if $\mu \geq \rho$; and

(b) $\lambda = \left(1 - \frac{1}{3} (\pi^\dagger)^2 \right)$, if $\mu < \rho$.

Theorem 8 (Main Proposition in Special Cases) (a) Under Assumption 1-4 and Condition (10), we have:

$$\mathcal{W}_{j,m} \Rightarrow \mathcal{W}_{\tau,\mu}, \quad (11)$$

where $\mathcal{W}_{\tau,\mu}$ is: (i) Recursive estimation:

$$\mathcal{W}_{\tau,\mu} = \mu^{-1} [\mathcal{B}_\ell(\tau - \rho) - \mathcal{B}_\ell(\tau - \rho - \mu)]' [\mathcal{B}_\ell(\tau - \rho) - \mathcal{B}_\ell(\tau - \rho - \mu)], \quad (12)$$

(ii) Rolling estimation:

$$\begin{aligned} \mathcal{W}_{\tau,\mu} = & \mu^{-1} \left\{ \left(\frac{2}{3\pi^\dagger} \right) \cdot 1(\mu \geq \rho) + \left(1 - \frac{1}{3} (\pi^\dagger)^2 \right) \cdot 1(\mu < \rho) \right\}^{-1} \times \quad (13) \\ & \left[\left\{ \mathcal{B}_\ell \left((\tau - \rho) \left(1 - \frac{(\tau - \rho)^2}{3\rho^2} \right) \right) - \mathcal{B}_\ell \left((\tau - \mu - \rho) \left(1 - \frac{(\tau - \mu - \rho)^2}{3\rho^2} \right) \right) \right\} \cdot 1(\mu + \rho \leq \tau < 2\rho) \right. \\ & + \left. \left\{ \mathcal{B}_\ell \left(\frac{2}{3}\rho \right) - \mathcal{B}_\ell \left((\tau - \mu - \rho) \left(1 - \frac{(\tau - \mu - \rho)^2}{3\rho^2} \right) \right) \right\} \cdot 1(2\rho < \tau \leq 2\rho + \mu) \right]' \\ & \times \left[\left\{ \mathcal{B}_\ell \left((\tau - \rho) \left(1 - \frac{(\tau - \rho)^2}{3\rho^2} \right) \right) - \mathcal{B}_\ell \left((\tau - \mu - \rho) \left(1 - \frac{(\tau - \mu - \rho)^2}{3\rho^2} \right) \right) \right\} \cdot 1(\mu + \rho \leq \tau < 2\rho) \right. \\ & + \left. \left\{ \mathcal{B}_\ell \left(\frac{2}{3}\rho \right) - \mathcal{B}_\ell \left((\tau - \mu - \rho) \left(1 - \frac{(\tau - \mu - \rho)^2}{3\rho^2} \right) \right) \right\} \cdot 1(2\rho < \tau \leq 2\rho + \mu) \right]. \end{aligned}$$

(b) Furthermore, under Assumptions 1-4 and condition $F = 0$, eq. (11) holds with

$$\mathcal{W}_{\tau,\mu} = \mu^{-1} [\mathcal{B}_\ell(\tau - \rho) - \mathcal{B}_\ell(\tau - \rho - \mu)]' [\mathcal{B}_\ell(\tau - \rho) - \mathcal{B}_\ell(\tau - \rho - \mu)]. \quad (14)$$

We reject the null hypothesis:

$$H_0 : \theta_j = \theta_0, \theta_0 = 0 \text{ for all } j = R + m, \dots, T \quad (15)$$

if $\max_{j \in \{R+m, \dots, T\}} W_{j,m} > \kappa_{\alpha,\ell}$, where $\kappa_{\alpha,\ell}$ are the critical values at the 100 α % significance level and are reported for $\alpha = 0.05$ in Table 1, Panel A for eq. (12) and (14) for various values of $\mu = m/T$ and number of restrictions, ℓ ; and in Table 1, Panel B for eq. (13) for various combinations of μ, ρ, ℓ

2 Proofs

Proof of Equation (1). Note that

$$\begin{aligned}\widehat{\theta}_j &= \left(m^{-1} \sum_{t=j-m+1}^j \widehat{g}_t \widehat{g}'_t \right)^{-1} \left(m^{-1} \sum_{t=j-m+1}^j \widehat{g}_t v_{t+h}(\widehat{\gamma}_{t,R}) \right) \\ &= \left(m^{-1} \sum_{t=j-m+1}^j \widehat{g}_t \widehat{g}'_t \right)^{-1} \left(m^{-1} \sum_{t=j-m+1}^j f_{t+h}(\widehat{\gamma}_{t,R}) \right).\end{aligned}\quad (16)$$

From a mean value expansion of $f_{t+h}(\widehat{\gamma}_{t,R})$ around γ^* we have:

$$f_{t+h}(\widehat{\gamma}_{t,R}) = f_{t+h} + f_{t+h,\gamma}(\widehat{\gamma}_{t,R} - \gamma^*) + w_{t+h},$$

where w_{t+h} is the remainder. Furthermore, by Assumption 1(i),

$$\begin{aligned}m^{-1/2} \sum_{t=j-m+1}^j f_{t+h}(\widehat{\gamma}_{t,R}) &= m^{-1/2} \sum_{t=j-m+1}^j f_{t+h} + m^{-1/2} \sum_{t=j-m+1}^j f_{t+h,\gamma} B_t H_t \\ &\quad + m^{-1/2} \sum_{t=j-m+1}^j w_{t+h},\end{aligned}\quad (17)$$

As in the proof of equation (4.1) in West (1996), note that

$$\begin{aligned}m^{-1/2} \sum_{t=j-m+1}^j f_{t+h,\gamma} B_t H_t &= m^{-1/2} F B \sum_{t=j-m+1}^j H_t + \widetilde{A}_j, \text{ where} \\ \widetilde{A}_j &\equiv m^{-1/2} \sum_{t=j-m+1}^j (f_{t+h,\gamma} - F) B H_t + m^{-1/2} F \sum_{t=j-m+1}^j (B_t - B) H_t \\ &\quad + m^{-1/2} \sum_{t=j-m+1}^j (f_{t+h,\gamma} - F) (B_t - B) H_t.\end{aligned}\quad (18)$$

Assumption 2 implies that the last three terms in \widetilde{A}_j are $o_p(1)$.

Therefore, by equations (16), (17) and (18), we have:

$$\begin{aligned}m^{1/2} \widehat{\theta}_j &= \left(m^{-1} \sum_{t=j-m+1}^j \widehat{g}_t \widehat{g}'_t \right)^{-1} \left(m^{-1/2} \sum_{t=j-m+1}^j f_{t+h}(\widehat{\gamma}_{t,R}) \right) \\ &= \left(m^{-1} \sum_{t=j-m+1}^j \widehat{g}_t \widehat{g}'_t \right)^{-1} \left(m^{-1/2} \sum_{t=j-m+1}^j f_{t+h} + m^{-1/2} F B \sum_{t=j-m+1}^j H_t + \widetilde{A}_j + m^{-1/2} \sum_{t=j-m+1}^j w_{t+h} \right) \\ &= \left(m^{-1} \sum_{t=j-m+1}^j \widehat{g}_t \widehat{g}'_t \right)^{-1} [I_\ell, F B] \left\{ m^{-1/2} \sum_{t=j-m+1}^j \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} \right\} + A_j^\dagger,\end{aligned}$$

where

$$A_j^\dagger \equiv \left(m^{-1} \sum_{t=j-m+1}^j \widehat{g}_t \widehat{g}'_t \right)^{-1} \left[\widetilde{A}_j + m^{-1/2} \sum_{t=j-m+1}^j w_{t+h} \right].$$

Thus,

$$\begin{aligned} m^{1/2} \widehat{\theta}_j &= \left(m^{-1} \sum_{t=j-m+1}^j \widehat{g}_t \widehat{g}'_t \right)^{-1} [I_\ell, FB] \left(\frac{T}{m} \right)^{1/2} \left\{ \frac{1}{T^{1/2}} \sum_{t=R}^j \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \frac{1}{T^{1/2}} \sum_{t=R}^{j-m} \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} \right\} + A_j^\dagger \\ &= G^{-1} [I_\ell, FB] \left(\frac{T}{m} \right)^{1/2} \frac{1}{T^{1/2}} \left\{ \sum_{t=R}^j \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \sum_{t=R}^{j-m} \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} \right\} + A_j, \text{ where} \end{aligned}$$

$$A_j = A_j^\dagger + \left\{ \left(m^{-1} \sum_{t=j-m+1}^j \widehat{g}_t \widehat{g}'_t \right)^{-1} - G^{-1} \right\} [I_\ell, FB] \left(\frac{T}{m} \right)^{1/2} \left\{ \frac{1}{T^{1/2}} \sum_{t=R}^j \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \frac{1}{T^{1/2}} \sum_{t=R}^{j-m} \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} \right\}.$$

By Assumption 2(v) and arguments similar to West (1996, proof of equation 4.1) and West and McCracken (1998, Lemma 4.1), $\lim_{T \rightarrow \infty} \sup_j \left| m^{-1/2} \sum_{t=j-m+1}^j w_{t+h} \right| = o_p(1)$. In addition, from arguments similar to those in Lemma 4.3 in West and McCracken (1998), Assumption 2 and consistency of $\widehat{\gamma}_{t,R}$, $\lim_{T \rightarrow \infty} \sup_j \left(\left(m^{-1} \sum_{t=j-m+1}^j \widehat{g}_t \widehat{g}'_t \right)^{-1} - G^{-1} \right) = 0$. Therefore, Assumptions 1 and 2 ensure that $\lim_{T \rightarrow \infty} \sup_j A_j = o_p(1)$. ■

Proof of Proposition 1. By Hansen (1992), under Assumptions 1-4 and $T^{-1/2} \xi_T \rightarrow \xi$ in $\mathcal{D}_{\mathbb{R}^{\ell+q}}[0, 1]$ then

$$\frac{1}{\sqrt{T}} \sum_{t=1}^j \begin{pmatrix} b_{R,t,j} \cdot I_\ell & 0 \\ 0 & a_{R,t,j} \cdot I_q \end{pmatrix} \begin{pmatrix} f_{t+h} \\ h_t \end{pmatrix} - C_T^*(\tau) \Rightarrow \int_0^\tau \begin{pmatrix} \sigma_f(s) \cdot I_\ell & 0 \\ 0 & \sigma_h(s, \tau) \cdot I_q \end{pmatrix} d\xi(s),$$

where $\xi(s) = S^{1/2} \mathcal{B}_{t+q}(s)$, $z_t = \sum_{k=1}^\infty E_t \left([f_{t+h+k} \quad h_{t+k}]' \right)$ and

$$\begin{aligned} C_T^*(\tau) &= \left\{ T^{-1/2} \sum_{t=1}^{\lceil \tau T \rceil} \left[\begin{pmatrix} b_{R,t,j} \cdot I_\ell & 0 \\ 0 & a_{R,t,j} \cdot I_q \end{pmatrix} - \begin{pmatrix} b_{R,t-1,j} \cdot I_\ell & 0 \\ 0 & a_{R,t-1,j} \cdot I_q \end{pmatrix} \right] z_t \right. \\ &\quad \left. - T^{-1/2} \begin{pmatrix} b_{R,j,j} \cdot I_\ell & 0 \\ 0 & a_{R,j,j} \cdot I_q \end{pmatrix} z_{j+1} \right\}. \end{aligned}$$

The proof follows from the fact that $\sup_\tau C_T^*(\tau) = o_p(1)$, using the same reasoning as in Cavaliere (2004, Proof of Theorem 4), and the fact that the variances $\sigma_f(s), \sigma_h(s, \tau)$ are square integrable and bounded. ■

Proof of Proposition 2. It follows directly from Proposition 1 and Assumption 2 that

$$T^{-1/2} \sum_{t=R}^j \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} \Rightarrow \int_0^\tau \Omega(s, \tau)^{1/2} S^{1/2} d\mathcal{B}_{l+q}(s).$$

Thus,

$$\begin{aligned} m^{1/2} \hat{\theta}_j &= G^{-1} \left(\frac{T}{m} \right)^{1/2} [I_\ell, FB] \left(\frac{1}{\sqrt{T}} \sum_{t=R}^j \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \frac{1}{\sqrt{T}} \sum_{t=R}^{j-m} \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} \right) + A_j \Rightarrow \\ &\mu^{-1/2} G^{-1} [I_\ell, FB] \left(\int_0^\tau \Omega(s, \tau)^{1/2} S^{1/2} d\mathcal{B}_{l+q}(s) - \int_0^{\tau-\mu} \Omega(s, \tau - \mu)^{1/2} S^{1/2} d\mathcal{B}_{l+q}(s) \right) \quad (19) \\ &= \mu^{-1/2} G^{-1} [I_\ell, FB] \left(\begin{aligned} &\int_0^{\tau-\mu} \left[\Omega(s, \tau)^{1/2} - \Omega(s, \tau - \mu)^{1/2} \right] S^{1/2} d\mathcal{B}_{l+q}(s) \\ &+ \int_{\tau-\mu}^\tau \Omega(s, \tau)^{1/2} S^{1/2} d\mathcal{B}_{l+q}(s) \end{aligned} \right) \\ &= \mu^{-1/2} G^{-1} [I_\ell, FB] \int_0^\tau \left(\begin{aligned} &\left[\Omega(s, \tau)^{1/2} - \Omega(s, \tau - \mu)^{1/2} \right] \cdot \mathbf{1}(s \leq \tau - \mu) \\ &+ \Omega(s, \tau)^{1/2} \cdot \mathbf{1}(\tau - \mu < s \leq \tau) \end{aligned} \right) S^{1/2} d\mathcal{B}_{l+q}(s) \\ &= \int_0^\tau \omega(s, \tau) d\mathcal{B}_{l+q}(s) = \mathcal{B}_l \left(\int_0^\tau \omega(s, \tau) \omega(s, \tau)' ds \right), \end{aligned}$$

where $\omega(s, \tau)$, $\tilde{\omega}(s, \tau)$ are defined in Proposition 2. The second line follows from Assumptions 2 and 3 as well as Proposition 1; the last equality follows from Lemma 2 in Cavaliere (2004). ■

Proof of Proposition 3. Note that:

$$\begin{aligned} \omega(s, \tau) &= \mu^{-\frac{1}{2}} G^{-1} [I_\ell, FB] \left[\begin{aligned} &\left(\Omega(s, \tau)^{1/2} - \Omega(s, \tau - \mu) \right) \cdot \mathbf{1}(s \leq \tau - \mu) \\ &+ \Omega(s, \tau)^{1/2} \cdot \mathbf{1}(\tau - \mu < s \leq \tau) \end{aligned} \right] S^{1/2} \\ &= \mu^{-\frac{1}{2}} G^{-1} [I_\ell, FB] \left[\begin{aligned} &\sigma_f(s) \cdot \mathbf{1}(\tau - \mu \leq s < \tau) \cdot I_\ell && 0 \\ &0 && (\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu)) \cdot \mathbf{1}(s \leq \tau - \mu) \\ &&& + \sigma_h(s, \tau) \cdot \mathbf{1}(\tau - \mu \leq s \leq \tau) \cdot I_q \end{aligned} \right] S^{1/2} \\ (i) \int_{\tau-\mu}^\tau \sigma_f^2(s) ds &= \int_{\tau-\mu}^\tau (\mathbf{1}(s \geq \rho))^2 ds = \int_{\tau-\mu}^\tau ds = \mu; \end{aligned}$$

(ii) Recursive case:¹ Let $\tilde{\pi} \equiv \mu / (\tau - \mu)$.

$$\begin{aligned}
\int_{\tau-\mu}^{\tau} \sigma_f(s) \sigma_h(s, \tau) ds &= \int_{\tau-\mu}^{\tau} 1(s \geq \rho) \cdot ([\ln(\tau) - \ln(\rho)] \cdot 1(s \leq \rho) + [\ln(\tau) - \ln(s)] \cdot 1(s > \rho)) ds \\
&= \int_{\tau-\mu}^{\tau} [\ln(\tau) - \ln(s)] ds = \int_{\tau-\mu}^{\tau} \ln(\tau) ds - \int_{\tau-\mu}^{\tau} \ln(s) ds \\
&= \ln(\tau) (\tau - \tau + \mu) - (\ln(\tau)\tau - \tau) + (\ln(\tau - \mu)(\tau - \mu) - (\tau - \mu)) \\
&= -\ln(\tau) (\tau - \mu) + \tau + \ln(\tau - \mu)(\tau - \mu) - \tau + \mu \\
&= \mu - (\tau - \mu) \ln\left(\frac{\tau}{\tau - \mu}\right) = \mu [1 - \tilde{\pi}^{-1} \ln(1 + \tilde{\pi})],
\end{aligned}$$

Furthermore,

$$\int_0^{\tau-\mu} (\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 ds = \int_0^{\tau-\mu} [\ln(\tau) - \ln(\tau - \mu)]^2 ds = (\tau - \mu) \left[\ln\left(\frac{\tau}{\tau - \mu}\right) \right]^2.$$

$$\begin{aligned}
\int_{\tau-\mu}^{\tau} \sigma_h^2(s, \tau) ds &= \int_{\tau-\mu}^{\tau} ([\ln(\tau) - \ln(\rho)] \cdot 1(s < \rho) + [\ln(\tau) - \ln(s)] \cdot 1(s \geq \rho))^2 ds \\
&= \int_{\tau-\mu}^{\tau} [\ln(\tau) - \ln(s)]^2 ds = \int_{\tau-\mu}^{\tau} (\ln(\tau)^2 - 2\ln(\tau)\ln(s) + \ln(s)^2) ds \\
&= \ln(\tau)^2 \mu - 2\ln(\tau) [\ln(\tau)\tau - \tau - \ln(\tau - \mu)(\tau - \mu) + \tau - \mu] + \\
&\quad + \ln(\tau)^2 \tau - 2\tau \ln(\tau) + 2\tau - \ln(\tau - \mu)^2 (\tau - \mu) + 2(\tau - \mu) \ln(\tau - \mu) - 2(\tau - \mu) \\
&= \ln(\tau)^2 \mu - \ln(\tau)^2 \tau + 2\ln(\tau) \ln(\tau - \mu) (\tau - \mu) \\
&\quad + 2\ln(\tau) \mu - 2\tau \ln(\tau) - \ln(\tau - \mu)^2 (\tau - \mu) + 2(\tau - \mu) \ln(\tau - \mu) + 2\mu \\
&= 2\mu + \ln(\tau)^2 (\mu - \tau) - \ln(\tau - \mu)^2 (\tau - \mu) + 2\ln(\tau) \ln(\tau - \mu) (\tau - \mu) \\
&\quad + 2(\tau - \mu) \ln(\tau - \mu) + 2\ln(\tau) (\mu - \tau) \\
&= 2\mu + \ln(\tau)^2 (\mu - \tau) - \ln(\tau - \mu)^2 (\tau - \mu) \\
&\quad + 2\ln(\tau) \ln(\tau - \mu) (\tau - \mu) + 2(\tau - \mu) \ln\left(\frac{\tau - \mu}{\tau}\right) \\
&= 2\mu - 2(\tau - \mu) \ln\left(\frac{\tau}{\tau - \mu}\right) - (\tau - \mu) \left[\ln\left(\frac{\tau}{\tau - \mu}\right) \right]^2
\end{aligned}$$

¹Note $\int \ln(x) dx = \ln(x)x - x + c$; $\int \ln(x)^2 dx = x \ln(x)^2 - 2x \ln(x) + 2x + c$

$$\begin{aligned}
& \int_0^\tau [(\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 \cdot 1(s \leq \tau - \mu) + \sigma_h^2(s, \tau) \cdot 1(\tau - \mu < s < \tau)] ds \\
&= \int_0^{\tau - \mu} (\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 ds + \int_{\tau - \mu}^\tau \sigma_h^2(s, \tau) ds \\
&= (\tau - \mu) \left[\ln \left(\frac{\tau}{\tau - \mu} \right) \right]^2 + 2\mu - 2(\tau - \mu) \ln \left(\frac{\tau}{\tau - \mu} \right) - (\tau - \mu) \left[\ln \left(\frac{\tau}{\tau - \mu} \right) \right]^2 \\
&= 2\mu - 2(\tau - \mu) \ln \left(\frac{\tau}{\tau - \mu} \right) = 2\mu [1 - \tilde{\pi}^{-1} \ln(1 + \tilde{\pi})]
\end{aligned}$$

(iii) Rolling case: Let $\pi^\dagger \equiv \frac{\mu}{\rho}$.

In the rolling estimation scheme there are two possible cases. Case (a) occurs when $\tau - \rho \geq \rho$, while (b) when $\tau - \rho < \rho$. We consider the calculation of the respective integrals in these two cases. We show that the covariance is the same in both cases, no matter whether $\mu \geq \rho$ or $\mu < \rho$.

Case (a): $\tau - \rho \geq \rho$.

This allows for two sub-cases: (i) $\mu \geq \rho \Leftrightarrow \tau - \rho \geq \tau - \mu \geq \rho$ and (ii) $\mu < \rho \Leftrightarrow \tau - \mu > \tau - \rho \geq \rho$ (recall that $\tau \geq \rho + \mu$).

In case (i),

$$\begin{aligned}
\int_{\tau - \mu}^\tau \sigma_h(s, \tau) \sigma_f(s) ds &= \int_{\tau - \mu}^\tau 1(s \geq \rho) \cdot \frac{1}{\rho} \left[\begin{array}{l} s \cdot 1(s \leq \rho) + \rho \cdot 1(\rho < s \leq \tau - \rho) \\ + (\tau - s) \cdot 1(s > \tau - \rho) \end{array} \right] ds = \\
&= \int_{\tau - \mu}^\tau \frac{1}{\rho} [\rho \cdot 1(\rho \leq s \leq \tau - \rho) + (\tau - s) \cdot 1(s > \tau - \rho)] ds = \\
&= (\mu - \rho) + \frac{1}{2}\rho = \mu - \frac{1}{2}\rho = \mu \left(1 - \frac{1}{2} \frac{\rho}{\mu} \right) = \mu \left(1 - \frac{1}{2\pi^\dagger} \right);
\end{aligned}$$

whereas in case (ii),

$$\begin{aligned}
\int_{\tau - \mu}^\tau \sigma_h(s, \tau) \sigma_f(s) ds &= \int_{\tau - \mu}^\tau 1(s \geq \rho) \cdot \frac{1}{\rho} \left[\begin{array}{l} s \cdot 1(s \leq \rho) + \rho \cdot 1(\rho < s \leq \tau - \rho) \\ + (\tau - s) \cdot 1(s > \tau - \rho) \end{array} \right] ds = \\
&= \int_{\tau - \mu}^\tau \frac{1}{\rho} [(\tau - s) \cdot 1(s > \tau - \rho)] ds = \int_{\tau - \mu}^\tau \frac{1}{\rho} (\tau - s) ds = \frac{1}{2} \frac{\mu^2}{\rho} = \frac{1}{2} \mu \pi^\dagger.
\end{aligned}$$

Furthermore,

$$\begin{aligned} & \int_0^{\tau-\mu} (\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 ds = \\ & \left(\frac{1}{\rho} \right)^2 \int_0^{\tau-\mu} \left(\begin{array}{l} [s \cdot 1(s \leq \rho) + \rho \cdot 1(\rho < s \leq \tau - \rho) + (\tau - s) \cdot 1(\tau - \rho < s \leq \tau)] - \\ [s \cdot 1(s \leq \rho) + \rho \cdot 1(\rho < s \leq \tau - \mu - \rho) + \\ (\tau - \mu - s) \cdot 1(\tau - \mu - \rho < s \leq \tau - \mu)] \end{array} \right)^2 ds = \\ & \left(\frac{1}{\rho} \right)^2 \int_0^{\tau-\mu} \left(\begin{array}{l} \rho \cdot 1(\tau - \mu - \rho \leq s \leq \tau - \rho) + (\tau - s) \cdot 1(\tau - \rho < s \leq \tau) \\ -(\tau - \mu - s) \cdot 1(\tau - \mu - \rho < s \leq \tau - \mu) \end{array} \right)^2 ds \end{aligned}$$

The expression above simplifies:

(i) $\mu \geq \rho \Leftrightarrow \tau - \rho > \tau - \mu \geq \rho$

$$\int_0^{\tau-\mu} (\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 ds = \left(\frac{1}{\rho} \right)^2 \int_{\tau-\mu-\rho}^{\tau-\mu} (\rho - (\tau - \mu - s))^2 ds = \frac{1}{3}\rho.$$

In addition,

$$\begin{aligned} \int_{\tau-\mu}^{\tau} \sigma_h^2(s, \tau) ds &= \left(\frac{1}{\rho} \right)^2 \int_{\tau-\mu}^{\tau} [s \cdot 1(s \leq \rho) + \rho \cdot 1(\rho < s \leq \tau - \rho) + (\tau - s) \cdot 1(s > \tau - \rho)]^2 ds = \\ &= \left(\frac{1}{\rho} \right)^2 \left(\int_{\tau-\mu}^{\tau-\rho} \rho^2 ds + \int_{\tau-\rho}^{\tau} (\tau - s)^2 ds \right) = (\mu - \rho) + \frac{1}{3}\rho = \mu - \frac{2}{3}\rho. \end{aligned}$$

Thus,

$$\int_0^{\tau-\mu} (\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 ds + \int_{\tau-\mu}^{\tau} \sigma_h^2(s) ds = \mu - \frac{1}{3}\rho = \mu \left(1 - \frac{1}{3\pi^\dagger} \right).$$

(ii) $\mu < \rho \Leftrightarrow \tau - \mu > \tau - \rho \geq \rho$

$$\begin{aligned} & \int_0^{\tau-\mu} (\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 ds = \\ & \left(\frac{1}{\rho} \right)^2 \int_0^{\tau-\mu} \left(\begin{array}{l} (\rho - (\tau - \mu - s)) \cdot 1(\tau - \mu - \rho \leq s \leq \tau - \rho) + (\tau - s) \cdot 1(\tau - \rho < s < \tau) \\ -(\tau - \mu - s) \cdot 1(\tau - \mu > s > \tau - \rho) \end{array} \right)^2 ds \\ & \left(\frac{1}{\rho} \right)^2 \int_0^{\tau-\mu} [(\rho - (\tau - \mu - s)) \cdot 1(\tau - \mu - \rho \leq s \leq \tau - \rho) + \mu \cdot 1(\tau - \mu > s > \tau - \rho)]^2 ds \\ & \left(\frac{1}{\rho} \right)^2 \int_{\tau-\mu-\rho}^{\tau-\rho} [\rho - (\tau - \mu - s)]^2 ds + \left(\frac{\mu}{\rho} \right)^2 \int_{\tau-\rho}^{\tau-\mu} ds = -\frac{1}{3} \frac{\mu^2}{\rho^2} (2\mu - 3\rho) = \frac{\mu^2}{\rho} - \frac{2}{3} \frac{\mu^3}{\rho^2} \end{aligned}$$

In addition,

$$\begin{aligned}\int_{\tau-\mu}^{\tau} \sigma_h^2(s, \tau) ds &= \left(\frac{1}{\rho}\right)^2 \int_{\tau-\mu}^{\tau} [s \cdot 1(s \leq \rho) + \rho \cdot 1(\rho < s \leq \tau - \rho) + (\tau - s) \cdot 1(s > \tau - \rho)]^2 ds = \\ &= \left(\frac{1}{\rho}\right)^2 \int_{\tau-\mu}^{\tau} (\tau - s)^2 ds = \frac{1}{3} \frac{\mu^3}{\rho^2}\end{aligned}$$

Thus,

$$\int_0^{\tau-\mu} (\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 ds + \int_{\tau-\mu}^{\tau} \sigma_h^2(s) ds = \frac{\mu^2}{\rho} - \frac{2\mu^3}{3\rho^2} + \frac{1}{3} \frac{\mu^3}{\rho^2} = \mu\pi^\dagger \left(1 - \frac{1}{3\pi^\dagger}\right).$$

Case (b): $\tau - \rho < \rho$.

Note that, since $\tau \geq \rho + \mu$, in this case the only possible subcase is $\mu < \rho$. Thus,

$$\begin{aligned}\int_{\tau-\mu}^{\tau} \sigma_f(s) \sigma_h(s, \tau) ds &= \frac{1}{\rho} \int_{\tau-\mu}^{\tau} 1(s \geq \rho) \cdot \left[\begin{array}{l} s \cdot 1(s < \tau - \rho) + (\tau - \rho) \cdot 1(\tau - \rho \leq s \leq \rho) \\ + (\tau - s) \cdot 1(s > \rho) \end{array} \right] ds = \\ &= \frac{1}{\rho} \int_{\tau-\mu}^{\tau} (\tau - s) \cdot 1(s > \rho) ds = \frac{1}{\rho} \int_{\tau-\mu}^{\tau} (\tau - s) ds = \frac{1}{2} \frac{\mu^2}{\rho} = \frac{1}{2} \mu\pi^\dagger\end{aligned}$$

Furthermore,

$$\begin{aligned}&\int_0^{\tau-\mu} (\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 ds \\ &= \left(\frac{1}{\rho}\right)^2 \int_0^{\tau-\mu} \left(\begin{array}{l} s \cdot 1(s < \tau - \rho) + (\tau - \rho) \cdot 1(\tau - \rho \leq s \leq \rho) + (\tau - s) \cdot 1(s > \rho) - \\ \left(s \cdot 1(s < \tau - \mu - \rho) + (\tau - \mu - \rho) \cdot 1(\tau - \mu - \rho \leq s \leq \rho) \right) \\ + (\tau - \mu - s) \cdot 1(s > \rho) \end{array} \right)^2 ds \\ &= \left(\frac{1}{\rho}\right)^2 \int_0^{\tau-\mu} \left(\begin{array}{l} (s - (\tau - \mu - \rho)) \cdot 1(\tau - \mu - \rho < s < \tau - \rho) + \\ ((\tau - \rho) - (\tau - \mu - \rho)) \cdot 1(\tau - \rho \leq s \leq \rho) + (\tau - s - (\tau - \mu - s)) \cdot 1(s > \rho) \end{array} \right)^2 ds \\ &= \left(\frac{1}{\rho}\right)^2 \left(\int_{\tau-\mu-\rho}^{\tau-\rho} (s - (\tau - \mu - \rho))^2 ds + \int_{\tau-\rho}^{\rho} \mu^2 ds + \int_{\rho}^{\tau-\mu} \mu^2 ds \right) \\ &= -\frac{1}{\rho^2} \left(\mu^2 (\mu - \tau + \rho) - \frac{1}{3} \mu^3 + \mu^2 (\tau - 2\rho) \right) = \left(\frac{\mu}{\rho}\right)^2 \left(\frac{1}{3} \mu - (\mu - \rho) \right)\end{aligned}$$

In addition, $\int_{\tau-\mu}^{\tau} \sigma_h^2(s, \tau) ds = \frac{1}{\rho^2} \int_{\tau-\mu}^{\tau} (\tau - s)^2 ds = \frac{1}{3} \frac{\mu^3}{\rho^2}$. Furthermore,

$$\int_0^{\tau-\mu} (\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 ds + \int_{\tau-\mu}^{\tau} \sigma_h^2(s) ds = \left(\frac{1}{\rho}\right)^2 \left(\frac{1}{3} \mu^3 - \mu^2 (\mu - \rho) \right) + \frac{1}{3} \frac{\mu^3}{\rho^2} = \mu\pi^\dagger \left(1 - \frac{1}{3\pi^\dagger}\right).$$

■

Proof of Proposition 4. From Proposition 2 (in particular, equation 19),

$$m^{1/2}\widehat{\theta}_j \Rightarrow \mu^{-\frac{1}{2}}G^{-1} [I_\ell, FB] \left(\int_0^\tau \Omega(s, \tau)^{1/2} S^{1/2} d\mathcal{B}_{l+q}(s) - \int_0^{\tau-\mu} \Omega(s, \tau - \mu)^{1/2} S^{1/2} d\mathcal{B}_{l+q}(s) \right).$$

By arguments similar to those in Proposition 2

$$m^{1/2}\widehat{\theta}_j \Rightarrow \mathcal{B} \left(\int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds \right) - \mathcal{B} \left(\int_0^{\tau-\mu} \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds \right)$$

where $\tilde{\omega}(s, \tau) = \mu^{-\frac{1}{2}}G^{-1} [I_\ell, FB] \Omega(s, \tau)^{1/2} S^{1/2}$.

Furthermore,

$$\begin{aligned} \int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds &= \mu^{-1}G^{-1} [I_\ell, FB] \int_0^\tau \Omega(s, \tau)^{1/2} S \Omega'(s, \tau)^{1/2} ds [I_\ell, FB]' G^{-1} = \\ &= \mu^{-1}G^{-1} [I_\ell, FB] \begin{bmatrix} \int_0^\tau \sigma_f^2(s) ds S_{ff} & \int_0^\tau \sigma_f(s) \sigma_h(s, \tau) ds S_{fh} \\ \int_0^\tau \sigma_f(s) \sigma_h(s, \tau) ds S'_{fh} & \int_0^\tau \sigma_h^2(s, \tau) ds S_{hh} \end{bmatrix} \begin{bmatrix} I_\ell \\ B' F' \end{bmatrix} G^{-1}. \end{aligned}$$

(i) $\int_0^\tau \sigma_f^2(s) ds = (\tau - \rho)$;

(ii) Recursive case:

$$\begin{aligned} \int_0^\tau \sigma_f(s) \sigma_h(s, \tau) ds &= \int_0^\tau 1(s \geq \rho) \cdot ([\ln(\tau) - \ln(\rho)] \cdot 1(s \leq \rho) + [\ln(\tau) - \ln(s)] \cdot 1(s > \rho)) ds \\ &= \int_\rho^\tau [\ln(\tau) - \ln(s)] ds = \int_\rho^\tau \ln(\tau) ds - \int_\rho^\tau \ln(s) ds = \\ &= \ln(\tau) (\tau - \rho) - (\ln(\tau)\tau - \tau) + (\ln(\rho)\rho - \rho) = \\ &= (\tau - \rho) - \rho \ln\left(\frac{\tau}{\rho}\right) = (\tau - \rho) \left(1 - \frac{\rho}{\tau - \rho} \ln\left(\frac{\tau}{\rho}\right)\right) \end{aligned}$$

$$\begin{aligned}
\int_0^\tau \sigma_h^2(s, \tau) ds &= \int_0^\tau ([\ln(\tau) - \ln(\rho)] \cdot 1(s < \rho) + [\ln(\tau) - \ln(s)] \cdot 1(s \geq \rho))^2 ds \\
&= \int_0^\rho [\ln(\tau) - \ln(\rho)]^2 ds + \int_\rho^\tau [\ln(\tau) - \ln(s)]^2 ds \\
&= \rho (\ln^2(\tau) - 2\ln(\tau)\ln(\rho) + \ln^2(\rho)) + (\tau - \rho) \ln^2(\tau) \\
&\quad - 2\ln\tau \int_\rho^\tau \ln(s) ds + \int_\rho^\tau \ln^2(s) ds \\
&= \rho (\ln^2(\tau) - 2\ln(\tau)\ln(\rho) + \ln^2(\rho)) + (\tau - \rho) \ln^2(\tau) - \\
&\quad - 2\ln(\tau) (\tau \ln(\tau) - \tau - \rho \ln(\rho) + \rho) + \\
&\quad + (\tau \ln^2(\tau) - 2\tau \ln(\tau) + 2\tau) - (\rho \ln^2(\rho) - 2\rho \ln(\rho) + 2\rho) \\
&= \rho \ln^2(\tau) - 2\rho \ln(\tau)\ln(\rho) + \rho \ln^2(\rho) + (\tau - \rho) \ln^2(\tau) - \\
&\quad - 2\tau \ln^2(\tau) + 2\tau \ln(\tau) + 2\rho \ln(\tau)\ln(\rho) - 2\rho \ln(\tau) + \\
&\quad + \tau \ln^2(\tau) - 2\tau \ln(\tau) + 2\tau - \rho \ln^2(\rho) + 2\rho \ln(\rho) - 2\rho \\
&= 2\tau - 2\rho - 2\rho \ln(\tau) + 2\rho \ln(\rho) = 2(\tau - \rho) \left(1 - \frac{\rho}{\tau - \rho} \ln\left(\frac{\tau}{\rho}\right) \right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^\tau \Omega(s, \tau)^{1/2} S \Omega'(s, \tau)^{1/2} ds &= \begin{bmatrix} \int_0^\tau \sigma_f^2(s) ds S_{ff} & \int_0^\tau \sigma_f(s) \sigma_h(s, \tau) ds S_{fh} \\ \int_0^\tau \sigma_f(s) \sigma_h(s, \tau) ds S'_{fh} & \int_0^\tau \sigma_h^2(s, \tau) ds S_{hh} \end{bmatrix} \\
&= (\tau - \rho) \begin{bmatrix} S_{ff} & \left(1 - \frac{\rho}{\tau - \rho} \ln\left(\frac{\tau}{\rho}\right)\right) S_{fh} \\ \left(1 - \frac{\rho}{\tau - \rho} \ln\left(\frac{\tau}{\rho}\right)\right) S'_{fh} & 2 \left(1 - \frac{\rho}{\tau - \rho} \ln\left(\frac{\tau}{\rho}\right)\right) S_{hh} \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds \\
&= \mu^{-1} G^{-1} [I_\ell, FB] \begin{bmatrix} \int_0^\tau \sigma_f^2(s) ds S_{ff} & \int_0^\tau \sigma_f(s) \sigma_h(s, \tau) ds S_{fh} \\ \int_0^\tau \sigma_f(s) \sigma_h(s, \tau) ds S'_{fh} & \int_0^\tau \sigma_h^2(s, \tau) ds S_{hh} \end{bmatrix} \begin{bmatrix} I_\ell \\ B'F' \end{bmatrix} G^{-1} \\
&= G^{-1} \frac{(\tau - \rho)}{\mu} [I_\ell, FB] \begin{bmatrix} S_{ff} & \left(1 - \frac{\rho}{\tau - \rho} \ln\left(\frac{\tau}{\rho}\right)\right) S_{fh} \\ \left(1 - \frac{\rho}{\tau - \rho} \ln\left(\frac{\tau}{\rho}\right)\right) S'_{fh} & 2 \left(1 - \frac{\rho}{\tau - \rho} \ln\left(\frac{\tau}{\rho}\right)\right) S_{hh} \end{bmatrix} \begin{bmatrix} I_\ell \\ B'F' \end{bmatrix} G^{-1} \\
&= \frac{(\tau - \rho)}{\mu} G^{-1} \left(S_{ff} + \left(1 - \rho \frac{\ln\left(\frac{\tau}{\rho}\right)}{\tau - \rho}\right) (S_{fh} B'F' + F B S'_{fh}) + 2 F B S_{hh} B'F' \left(1 - \rho \frac{\ln\left(\frac{\tau}{\rho}\right)}{\tau - \rho}\right) \right) G^{-1};
\end{aligned}$$

(iii) Rolling Case:

Case (a): $\tau - \rho \geq \rho$

$$\begin{aligned}
& \int_0^\tau \sigma_h(s, \tau) \sigma_f(s) ds \\
&= \int_0^\tau 1(s \geq \rho) \left(\frac{1}{\rho} [s \cdot 1(s \leq \rho) + \rho \cdot 1(\rho < s \leq \tau - \rho) + (\tau - s) \cdot 1(s > \tau - \rho)] \right) ds \\
&= \int_\rho^\tau \frac{1}{\rho} [\rho \cdot 1(\rho \leq s < \tau - \rho) + (\tau - s) \cdot 1(s > \tau - \rho)] ds \\
&= \frac{1}{\rho} \left(\int_\rho^{\tau-\rho} \rho ds + \int_{\tau-\rho}^\tau (\tau - s) ds \right) = \frac{1}{\rho} \left(\rho(\tau - 2\rho) + \tau\rho - \frac{1}{2}\tau^2 + \frac{1}{2}(\tau - \rho)^2 \right) \\
&= \tau - \frac{3}{2}\rho
\end{aligned}$$

$$\begin{aligned}
& \int_0^\tau \sigma_h^2(s, \tau) ds \\
&= \int_0^\tau \left(\frac{1}{\rho} \{s \cdot 1(s \leq \rho) + \rho \cdot 1(\rho < s \leq \tau - \rho) + (\tau - s) \cdot 1(s > \tau - \rho)\} \right)^2 ds \\
&= \left(\frac{1}{\rho} \right)^2 \left(\int_0^\rho s^2 \cdot 1(s \leq \rho) ds + \int_0^\tau \rho^2 \cdot 1(\rho < s \leq \tau - \rho) ds + \int_0^\tau (\tau - s)^2 \cdot 1(s > \tau - \rho) ds \right) \\
&= \left(\frac{1}{\rho} \right)^2 \left(\int_0^\rho s^2 ds + \rho^2 \int_\rho^{\tau-\rho} ds + \left(\frac{1}{\rho} \right)^2 \int_{\tau-\rho}^\tau (\tau - s)^2 ds \right) \\
&= \left(\frac{1}{\rho} \right)^2 \frac{1}{3} \rho^3 + (\tau - 2\rho) + \left(\frac{1}{\rho} \right)^2 \left(\tau^2 \rho + \frac{\tau^3 - (\tau - \rho)^3}{3} - 2\tau \frac{\tau^2 - (\tau - \rho)^2}{2} \right) \\
&= \tau - \frac{4}{3}\rho
\end{aligned}$$

Thus,

$$\int_0^\tau \Omega(s, \tau)^{1/2} S \Omega'(s, \tau)^{1/2} ds = \begin{bmatrix} (\tau - \rho) S_{ff} & (\tau - \frac{3}{2}\rho) S_{fh} \\ (\tau - \frac{3}{2}\rho) S'_{fh} & (\tau - \frac{4}{3}\rho) S_{hh} \end{bmatrix},$$

and

$$\begin{aligned}
& \int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds = \mu^{-1} G^{-1} [I_\ell, FB] \begin{bmatrix} (\tau - \rho) S_{ff} & (\tau - \frac{3}{2}\rho) S_{fh} \\ (\tau - \frac{3}{2}\rho) S'_{fh} & (\tau - \frac{4}{3}\rho) S_{hh} \end{bmatrix} \begin{bmatrix} I_\ell \\ B' F' \end{bmatrix} G^{-1} = \\
& \mu^{-1} G^{-1} \left(S_{ff}(\tau - \rho) + \left(\tau - \frac{3}{2}\rho \right) (FBS'_{fh} + S_{fh}B'F') + \left(\tau - \frac{4}{3}\rho \right) FBS_{hh}B'F' \right) G^{-1};
\end{aligned}$$

Case (b): $\tau - \rho < \rho$

$$\begin{aligned} & \int_0^\tau \sigma_f(s) \sigma_h(s, \tau) ds \\ &= \int_0^\tau \{1(s \geq \rho)\} \left\{ \frac{s}{\rho} \cdot 1(s \leq \tau - \rho) + \frac{(\tau - \rho)}{\rho} \cdot 1(\tau - \rho < s \leq \rho) + \frac{(\tau - s)}{\rho} \cdot 1(s > \rho) \right\} ds = \\ &= \int_0^\tau \frac{1}{\rho} (\tau - s) \cdot 1(s \geq \rho) ds = \frac{1}{\rho} \int_\rho^\tau (\tau - s) ds = \frac{1}{2\rho} (\tau - \rho)^2 \end{aligned}$$

$$\begin{aligned} \int_0^\tau \sigma_h^2(s) ds &= \int_0^\tau \left(\frac{1}{\rho} \{s \cdot 1(s \leq \tau - \rho) + (\tau - \rho) \cdot 1(\tau - \rho < s \leq \rho) + (\tau - s) \cdot 1(s > \rho)\} \right)^2 ds \\ &= \left(\frac{1}{\rho} \right)^2 \left(\int_0^{\tau - \rho} s^2 ds + \int_{\tau - \rho}^\rho (\tau - \rho)^2 ds + \int_\rho^\tau (\tau - s)^2 ds \right) \\ &= \left(\frac{1}{\rho} \right)^2 \left(\frac{1}{3} (\tau - \rho)^3 + (\tau - \rho)^2 (2\rho - \tau) + \tau^2 (\tau - \rho) + \frac{\tau^3 - \rho^3}{3} - \tau^3 + \tau\rho^2 \right) \\ &= \frac{1}{3\rho^2} (\tau - \rho)^2 (4\rho - \tau) \end{aligned}$$

Thus,

$$\int_0^\tau \Omega(s, \tau)^{1/2} S\Omega'(s, \tau)^{1/2} ds = \begin{bmatrix} (\tau - \rho)S_{ff} & \frac{1}{2\rho} (\tau - \rho)^2 S_{fh} \\ \frac{1}{2\rho} (\tau - \rho)^2 S'_{fh} & \frac{1}{3\rho^2} (\tau - \rho)^2 (4\rho - \tau) S_{hh} \end{bmatrix},$$

and

$$\begin{aligned} & \int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds \\ &= \mu^{-1} G^{-1} [I_\ell, FB] \begin{bmatrix} (\tau - \rho)S_{ff} & \frac{1}{2\rho} (\tau - \rho)^2 S_{fh} \\ \frac{1}{2\rho} (\tau - \rho)^2 S'_{fh} & \frac{(\tau - \rho)^2}{3\rho^2} (4\rho - \tau) S_{hh} \end{bmatrix} \begin{bmatrix} I_\ell \\ B'F' \end{bmatrix} G^{-1} \\ &= \frac{(\tau - \rho)}{\mu} G^{-1} \left\{ S_{ff} + \frac{(\tau - \rho)}{2\rho} (FBS'_{fh} + S_{fh}B'F') + \frac{(\tau - \rho)(4\rho - \tau)}{3\rho^2} FBS_{hh}F'B' \right\} G^{-1}. \end{aligned}$$

■

Proof of Theorem 5. The proof follows directly from Propositions 2 and 3. ■

Proof of Proposition 6. The result follows directly from Propositions 3 and 4 by imposing $F = 0$. ■

Proof of Proposition 7. From Proposition 2 note that:

$$\begin{aligned} \int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds &= \mu^{-1} G^{-1} \int_0^\tau [I_\ell, FB] \Omega(s, \tau)^{1/2} S\Omega(s, \tau)^{1/2} [I_\ell, FB]' ds G^{-1} \quad (20) \\ &= \mu^{-1} G^{-1} \int_0^\tau [\sigma_f^2(s) S_{ff} + \sigma_f(s) \sigma_h(s, \tau) (FBS_{hf} + S_{fh}B'F') + \sigma_h^2(s, \tau) FBS_{hh}B'F'] ds G^{-1} \end{aligned}$$

(i) Recursive case:

By imposing $-\frac{1}{2}(FBS_{hf} + S_{fh}B'F') = FBS_{hh}B'F'$ from condition (10) on equation (20), we can have:

$$\begin{aligned} \int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds &= \mu^{-1}G^{-1} \int_0^\tau \left[\begin{array}{c} \sigma_f^2(s) S_{ff} + \sigma_f(s) \sigma_h(s, \tau) (FBS_{hf} + S_{fh}B'F') \\ + \sigma_h^2(s, \tau) FBS_{hh}B'F' \end{array} \right] ds G^{-1} \\ &= \mu^{-1}G^{-1} \int_0^\tau \left[\begin{array}{c} \sigma_f^2(s) S_{ff} - 2\sigma_f(s) \sigma_h(s, \tau) FBS_{hh}B'F' \\ + \sigma_h^2(s, \tau) FBS_{hh}B'F' \end{array} \right] ds G^{-1}. \end{aligned}$$

From Proposition 4 note that $\int_0^\tau \sigma_h^2(s, \tau) ds = 2 \int_0^\tau \sigma_h(s, \tau) \sigma_f(s) ds$. This further simplifies the expression:

$$\begin{aligned} \int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds &= \mu^{-1}G^{-1} \int_0^\tau \left[\begin{array}{c} \sigma_f^2(s) S_{ff} - 2\sigma_f(s) \sigma_h(s, \tau) FBS_{hh}B'F' \\ + \sigma_h^2(s, \tau) FBS_{hh}B'F' \end{array} \right] ds G^{-1} \\ &= \mu^{-1}G^{-1} \left(\int_0^\tau \sigma_f^2(s) S_{ff} ds \right) G^{-1} = \frac{(\tau - \rho)}{\mu} G^{-1} S_{ff} G^{-1} \end{aligned}$$

(ii) Rolling case:

By imposing condition (10) on equation (20), we can further simplify:

$$\int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds = \mu^{-1}G^{-1} S_{ff} G^{-1} \int_0^\tau [\sigma_f(s)^2 - 2\sigma_f(s) \sigma_h(s, \tau) + \sigma_h^2(s, \tau)] ds. \quad (21)$$

Case (a): $\tau - \rho \geq \rho$.

By Proposition 4, equation (21) simplifies to:

$$\begin{aligned} \int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds &= \mu^{-1} \int_0^\tau [\sigma_f(s)^2 - 2\sigma_f(s) \sigma_h(s, \tau) + \sigma_h^2(s, \tau)] ds G^{-1} S_{ff} G^{-1} \quad (22) \\ &= \frac{2}{3} \frac{\rho}{\mu} G^{-1} S_{ff} G^{-1}, \end{aligned}$$

and is independent of τ .

Case (b): $\tau - \rho < \rho$.

By Proposition 4, equation (21) simplifies to:

$$\begin{aligned} \int_0^\tau \tilde{\omega}(s) \tilde{\omega}(s)' ds &= \frac{1}{\mu} \int_0^\tau [\sigma_f^2(s) - 2\sigma_f(s) \sigma_h(s, \tau) + \sigma_h^2(s, \tau)] ds G^{-1} S_{ff} G^{-1} \\ &= \frac{1}{\mu} \left[(\tau - \rho) - 2 \frac{1}{2\rho} (\tau - \rho)^2 + \frac{1}{3\rho^2} (\rho - \tau)^2 (4\rho - \tau) \right] G^{-1} S_{ff} G^{-1} \\ &= \left(\frac{\tau - \rho}{\mu} \right) \left(1 - \frac{(\tau - \rho)^2}{3\rho^2} \right) G^{-1} S_{ff} G^{-1}. \end{aligned} \quad (23)$$

Furthermore, from Proposition 2 we have that:

$$\begin{aligned}
& \int_0^\tau \omega(s, \tau) \omega(s, \tau)' ds = \mu^{-1} G^{-1} \int_0^\tau [I_\ell, FB] \left\{ \begin{array}{l} [\Omega(s, \tau)^{1/2} - \Omega(s, \tau - \mu)^{1/2}] \cdot 1(s \leq \tau - \mu) \\ + \Omega(s, \tau)^{1/2} \cdot 1(\tau - \mu < s \leq \tau) \end{array} \right\} \\
& \times S \left\{ \begin{array}{l} [\Omega(s, \tau)^{1/2} - \Omega(s, \tau - \mu)^{1/2}] \cdot 1(s \leq \tau - \mu) \\ + \Omega(s, \tau)^{1/2} \cdot 1(\tau - \mu < s \leq \tau) \end{array} \right\} [I_\ell, FB]' ds G^{-1} \\
& = \mu^{-1} G^{-1} \left\{ \begin{array}{l} \left(\int_{\tau-\mu}^\tau \sigma_f^2(s) ds \right) S_{ff} + \\ \left(\int_{\tau-\mu}^\tau \sigma_f(s) \sigma_h(s, \tau) ds \right) [FBS_{hf} + S_{fh}B'F'] + \\ \int_0^\tau \left[\begin{array}{l} (\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 \cdot 1(s \leq \tau - \mu) \\ + \sigma_h^2(s, \tau) \cdot 1(\tau - \mu < s \leq \tau) \end{array} \right] ds FBS_{hh}B'F' \end{array} \right\} G^{-1} \quad (24)
\end{aligned}$$

(i') Recursive case:

Similar to (i), by imposing $-\frac{1}{2}(FBS_{hf} + S_{fh}B'F') = FBS_{hh}B'F'$ from condition (10) and from Proposition 3 the fact that

$$\int_0^\tau \left[\begin{array}{l} (\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu))^2 \cdot 1(s \leq \tau - \mu) \\ + \sigma_h^2(s, \tau) \cdot 1(\tau - \mu < s \leq \tau) \end{array} \right] ds = 2 \int_{\tau-\mu}^\tau \sigma_f(s) \sigma_h(s, \tau) ds,$$

we can simplify equation (24) to:

$$\int_0^\tau \omega(s, \tau) \omega(s, \tau)' ds = \mu^{-1} G^{-1} \left(\int_{\tau-\mu}^\tau \sigma_f^2(s) ds \right) G^{-1} = \frac{\mu}{\mu} G^{-1} S_{ff} G^{-1} = G^{-1} S_{ff} G^{-1}.$$

(ii') Rolling case: Let $\pi^\dagger \equiv \frac{\mu}{\rho}$.

Further, under condition (10), we have:

$$\begin{aligned}
& \int_0^\tau \omega(s, \tau) \omega(s, \tau)' ds = \mu^{-1} G^{-1} S_{ff} G^{-1} \times \quad (25) \\
& \left[\int_{\tau-\mu}^\tau \sigma_f^2(s) ds - 2 \left(\int_{\tau-\mu}^\tau \sigma_h(s, \tau) \sigma_f(s) ds \right) + \int_0^\tau \left(\begin{array}{l} [\sigma_h(s, \tau) - \sigma_h(s, \tau - \mu)]^2 \cdot 1(s \leq \tau - \mu) \\ + \sigma_h^2(s, \tau) \cdot 1(\tau - \mu < s \leq \tau) \end{array} \right) ds \right].
\end{aligned}$$

By Proposition 3, equation (25) simplifies to:

Case (a): If $\mu \geq \rho$, then $\int_0^\tau \omega(s, \tau) \omega(s, \tau)' ds = \frac{2}{3\pi^\dagger}$;

Case (b): If $\mu < \rho$, then $\int_0^\tau \omega(s, \tau) \omega(s, \tau)' ds = 1 - \frac{1}{3} (\pi^\dagger)^2$; ■

Proof of Theorem 8. (a) From Proposition 2 and Theorem 5 we have

$$\begin{aligned}\mathcal{W}_{t,m} &= m\widehat{\theta}_j V_{\theta,\tau}^{-1} \widehat{\theta}_j \\ &\Rightarrow \left[\mathcal{B}_\ell \left(\int_0^\tau \widetilde{\omega}(s,\tau) \widetilde{\omega}(s,\tau)' ds \right) - \mathcal{B}_\ell \left(\int_0^{\tau-\mu} \widetilde{\omega}(s,\tau) \widetilde{\omega}(s,\tau)' ds \right) \right]' V_{\theta,\tau}^{-1} \times \\ &\times \left[\mathcal{B}_\ell \left(\int_0^\tau \widetilde{\omega}(s,\tau) \widetilde{\omega}(s,\tau)' ds \right) - \mathcal{B}_\ell \left(\int_0^{\tau-\mu} \widetilde{\omega}(s,\tau) \widetilde{\omega}(s,\tau)' ds \right) \right]\end{aligned}$$

Under condition (10) and by the results in Proposition 7:

(i) Recursive case:

$$V_{\theta,\tau} = V_\theta = G^{-1} S_{ff} G^{-1} \text{ and } \mathcal{B}_\ell \left(\int_0^\tau \widetilde{\omega}(s,\tau) \widetilde{\omega}(s,\tau)' ds \right) = (G^{-1} S_{ff} G^{-1})^{1/2} \mu^{-1/2} \mathcal{B}_\ell(\tau - \rho).$$

Thus,

$$\mathcal{W}_{t,m} \Rightarrow \mu^{-1} [\mathcal{B}_\ell(\tau - \rho) - \mathcal{B}_\ell(\tau - \mu - \rho)]' [\mathcal{B}_\ell(\tau - \rho) - \mathcal{B}_\ell(\tau - \mu - \rho)].$$

(ii) Rolling case:

$$\text{From Proposition 7, } V_{\theta,\tau} = V_\theta = (G^{-1} S_{ff} G^{-1}) \left[\left(\frac{2}{3\pi^\dagger} \right) \cdot 1(\mu \geq \rho) + \left(1 - \frac{1}{3} (\pi^\dagger)^2 \right) \cdot 1(\mu < \rho) \right].$$

Furthermore, when $\tau - \rho < \rho$,

$$\begin{aligned}\mathcal{B}_\ell \left(\int_0^\tau \widetilde{\omega}(s,\tau) \widetilde{\omega}(s,\tau)' ds \right) &= \mathcal{B}_\ell \left(\left(\frac{\tau - \rho}{\mu} \right) \left(1 - \frac{(\tau - \rho)^2}{3\rho^2} \right) G^{-1} S_{ff} G^{-1} \right) \\ &= (G^{-1} S_{ff} G^{-1})^{1/2} \mu^{-1/2} \mathcal{B}_\ell \left((\tau - \rho) \left(1 - \frac{(\tau - \rho)^2}{3\rho^2} \right) \right).\end{aligned}$$

Alternatively, when $\tau - \rho \geq \rho$,

$$\mathcal{B}_\ell \left(\int_0^\tau \widetilde{\omega}(s,\tau) \widetilde{\omega}(s,\tau)' ds \right) = (G^{-1} S_{ff} G^{-1})^{1/2} \mu^{-1/2} \mathcal{B}_\ell \left(\frac{2}{3} \rho \right).$$

Thus,

$$\begin{aligned}\mathcal{B}_\ell \left(\int_0^\tau \widetilde{\omega}(s,\tau) \widetilde{\omega}(s,\tau)' ds \right) \\ = (G^{-1} S_{ff} G^{-1})^{1/2} \mu^{-1/2} \left[\begin{array}{l} \mathcal{B}_\ell \left((\tau - \rho) \left(1 - \frac{(\tau - \rho)^2}{3\rho^2} \right) \right) \cdot 1(\mu + \rho \leq \tau < 2\rho) \\ + \mathcal{B}_\ell \left(\frac{2}{3} \rho \right) \cdot 1(\tau \geq 2\rho) \end{array} \right],\end{aligned}$$

$$\begin{aligned}\mathcal{B}_\ell \left(\int_0^{\tau-\mu} \widetilde{\omega}(s,\tau - \mu) \widetilde{\omega}(s,\tau - \mu)' ds \right) \\ = (G^{-1} S_{ff} G^{-1})^{1/2} \mu^{-1/2} \left[\begin{array}{l} \mathcal{B}_\ell \left((\tau - \mu - \rho) \left(1 - \frac{(\tau - \mu - \rho)^2}{3\rho^2} \right) \right) \cdot 1(\rho \leq \tau - \mu < 2\rho) \\ + \mathcal{B}_\ell \left(\frac{2}{3} \rho \right) \cdot 1(\tau - \mu \geq 2\rho) \end{array} \right].\end{aligned}$$

Consequently,

$$\begin{aligned}
& \mathcal{B}_\ell \left(\int_0^\tau \tilde{\omega}(s, \tau) \tilde{\omega}(s, \tau)' ds \right) - \mathcal{B}_\ell \left(\int_0^{\tau-\mu} \tilde{\omega}(s, \tau - \mu) \tilde{\omega}(s, \tau - \mu)' ds \right) \\
&= (G^{-1} S_{ff} G^{-1})^{1/2} \mu^{-1/2} \left[\left\{ \begin{array}{l} \mathcal{B}_\ell \left((\tau - \rho) \left(1 - \frac{(\tau - \rho)^2}{3\rho^2} \right) \right) \\ -\mathcal{B}_\ell \left((\tau - \mu - \rho) \left(1 - \frac{(\tau - \mu - \rho)^2}{3\rho^2} \right) \right) \end{array} \right\} \cdot 1(\mu + \rho \leq \tau < 2\rho) \right. \\
&\quad \left. + \left[\mathcal{B}_\ell \left(\frac{2}{3}\rho \right) - \mathcal{B}_\ell \left((\tau - \mu - \rho) \left(1 - \frac{(\tau - \mu - \rho)^2}{3\rho^2} \right) \right) \right] \cdot 1(2\rho \leq \tau < 2\rho + \mu) \right. \\
&\quad \left. + 0 \cdot 1(\tau \geq 2\rho + \mu) \right] \\
&\equiv \mathcal{B}_\ell \left(\int_0^\tau \omega(s, \tau) \omega(s, \tau)' ds \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{W}_{t,m} &\Rightarrow \mathcal{B}_\ell \left(\int_0^\tau \omega(s, \tau) \omega(s, \tau)' ds \right)' V_{\theta, \tau}^{-1} \mathcal{B}_\ell \left(\int_0^\tau \omega(s, \tau) \omega(s, \tau)' ds \right) = \\
&= \mu^{-1} \left\{ \left(\frac{2}{3\pi^\dagger} \right) \cdot 1(\mu \geq \rho) + \left(1 - \frac{1}{3} (\pi^\dagger)^2 \right) \cdot 1(\mu < \rho) \right\}^{-1} \times \\
&\quad \left[\left\{ \mathcal{B}_\ell \left((\tau - \rho) \left(1 - \frac{(\tau - \rho)^2}{3\rho^2} \right) \right) - \mathcal{B}_\ell \left((\tau - \mu - \rho) \left(1 - \frac{(\tau - \mu - \rho)^2}{3\rho^2} \right) \right) \right\} \cdot 1(\mu + \rho \leq \tau < 2\rho) \right. \\
&\quad \left. + \left\{ \mathcal{B}_\ell \left(\frac{2}{3}\rho \right) - \mathcal{B}_\ell \left((\tau - \mu - \rho) \left(1 - \frac{(\tau - \mu - \rho)^2}{3\rho^2} \right) \right) \right\} \cdot 1(2\rho < \tau \leq 2\rho + \mu) \right]' \\
&\quad \times \left[\left\{ \mathcal{B}_\ell \left((\tau - \rho) \left(1 - \frac{(\tau - \rho)^2}{3\rho^2} \right) \right) - \mathcal{B}_\ell \left((\tau - \mu - \rho) \left(1 - \frac{(\tau - \mu - \rho)^2}{3\rho^2} \right) \right) \right\} \cdot 1(\mu + \rho \leq \tau < 2\rho) \right. \\
&\quad \left. + \left\{ \mathcal{B}_\ell \left(\frac{2}{3}\rho \right) - \mathcal{B}_\ell \left((\tau - \mu - \rho) \left(1 - \frac{(\tau - \mu - \rho)^2}{3\rho^2} \right) \right) \right\} \cdot 1(2\rho < \tau \leq 2\rho + \mu) \right].
\end{aligned}$$

(b) Follows directly from Proposition 6 using the same arguments in the proof of (a). ■

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3 Additional Critical Value Tables

Table A.1a. Critical Values for the Fluctuation Rationality Test

Recursive Case

Panel A. 10% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	5.9556	5.4013	6.9650	6.4921
2	9.2245	8.5250	10.1145	9.7747
3	11.3230	10.2160	12.2866	11.7772
4	13.2317	12.1300	14.1436	13.4497
5	14.6695	13.6400	15.8784	15.3008
6	16.5875	15.5707	18.3062	17.8984
7	18.8531	17.6269	19.7706	19.4035
8	19.5394	18.2724	21.4959	20.6359
9	22.0456	20.9694	23.3407	22.6708
10	23.0327	21.7260	24.6531	23.7097

Panel B. 5% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	7.6103	6.8123	8.1529	8.0414
2	10.7828	10.3909	12.1409	11.1946
3	12.6497	11.8263	14.2097	13.1495
4	14.8763	14.2381	15.8727	15.4504
5	16.4838	16.1415	17.9421	17.4355
6	19.1056	18.1881	20.6415	19.9306
7	20.3082	19.5852	21.5059	20.9714
8	21.9336	20.6173	23.5220	22.9359
9	24.6251	23.6866	25.6793	25.6762
10	25.3684	24.1207	26.8021	25.7949

Panel C. 1% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	10.2989	9.6853	11.7167	10.7821
2	14.7120	14.8595	16.8443	16.0830
3	17.1015	16.0334	19.4287	16.9283
4	17.8342	17.2788	18.7771	18.4869
5	21.2392	20.5449	22.3047	23.3458
6	23.1487	22.8962	23.6097	23.9932
7	24.5539	23.6424	25.8011	26.0693
8	27.4389	25.7965	28.1082	27.6579
9	29.1371	27.9073	30.7444	28.9690
10	29.8898	28.5949	32.6835	29.7556

Note. The table reports the critical values, $\kappa_{\alpha, \ell}$, for several restrictions (ℓ) at $\alpha = 10\%$, 5% and 1% significance levels for $\max_{j \in \{R+m, \dots, T\}} \mathcal{W}_{j,m}$ for the recursive scheme under condition (10). Critical values are based on $T = 1000$ and 1000 Monte Carlo simulations; $\rho = R/T$ and $\mu = m/T$.

Table A.1b. Critical Values for the Fluctuation Rationality Test
Rolling Case

Panel A. 10% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	6.4740	5.5899	7.7008	7.2863
2	9.1746	8.5913	10.7956	10.6739
3	11.5663	10.8638	12.9363	12.3039
4	13.2258	12.6272	14.4234	13.9984
5	14.9127	14.6108	17.0396	16.5904
6	16.4976	15.6346	19.0793	18.3600
7	18.7578	17.7502	20.5095	19.8632
8	19.9329	19.2351	22.7822	22.0157
9	21.7496	21.2777	23.9299	23.9710
10	23.1398	22.2947	25.2720	24.8407

Panel B. 5% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	7.7122	6.9621	8.8102	8.5989
2	10.5702	10.0698	12.4778	12.1265
3	13.2956	12.3069	14.5513	13.9501
4	14.8771	14.2805	16.6307	15.6392
5	16.8451	16.6441	19.0969	18.6127
6	18.5144	17.5945	20.9080	20.1921
7	21.0426	19.7563	22.6405	21.9120
8	22.9293	21.4715	25.1263	24.2192
9	24.2818	23.4890	26.4981	26.1489
10	25.7621	24.6734	26.8059	26.7959

Panel C. 1% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	11.0943	10.3372	11.7440	11.0335
2	13.7842	14.1051	15.6558	15.7895
3	17.2460	16.4848	18.3441	18.0914
4	18.3709	18.0079	21.0888	21.4351
5	21.6826	20.6014	22.3863	22.5591
6	22.7820	22.9287	25.9931	25.1753
7	24.9451	23.4760	26.5961	25.3780
8	27.9311	26.4139	28.7270	28.1257
9	30.1262	28.8866	32.1298	30.3945
10	31.0603	30.4121	31.7719	32.4430

Note. The table reports the critical values, $\kappa_{\alpha, \ell}$, for several restrictions (ℓ) at $\alpha = 10\%$, 5% and 1% significance levels for $\max_{j \in \{R+m, \dots, T\}} \mathcal{W}_{j,m}$ for the rolling scheme under condition (10). Critical values are based on $T = 1000$ and 1000 Monte Carlo simulations; $\rho = R/T$ and $\mu = m/T$.

Table A.1c. Critical Values for the Fluctuation Rationality Test
Survey and Model-Free Forecasts
Panel A. 10% Significance Level

ℓ	$\tilde{\mu}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	10.0909	8.8274	7.7116	6.9555	6.4272	5.8410	4.9404	4.8508	4.0096
2	13.2456	11.4773	10.7955	9.6482	9.3648	8.3442	7.7478	7.4669	6.2243
3	15.9915	14.2049	13.3396	11.6461	11.4939	10.4839	9.3900	8.9699	7.9423
4	18.4447	15.5897	15.1254	13.8661	13.2415	12.7312	11.5331	10.7335	9.3509
5	19.9690	18.2447	16.7190	15.7116	15.0672	14.1355	13.1798	12.1317	11.4857
6	22.3413	20.3183	19.1924	18.0867	17.3395	15.6658	14.7640	13.4408	12.3823
7	24.3462	22.3182	21.0891	20.0822	18.4705	17.7375	16.8276	15.6473	13.9220
8	26.5930	23.8825	22.4070	21.6763	20.1078	18.7631	18.1456	17.0475	15.8832
9	27.8409	26.0117	24.5040	23.3912	21.4939	20.6265	19.2173	18.0560	17.2398
10	29.2933	27.1103	25.8483	24.6502	23.4647	22.5659	20.6670	20.0081	18.4020

Panel B. 5% Significance Level

ℓ	$\tilde{\mu}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	11.8290	10.5637	8.9252	8.1468	8.1409	7.2803	6.4978	6.0837	5.4695
2	14.9966	13.0846	12.8141	10.9084	11.1314	9.9386	9.1724	9.0589	7.8305
3	17.6768	15.7548	15.0608	13.4383	13.2113	12.6018	10.9597	10.8426	9.4727
4	19.8434	17.6051	17.0158	16.3186	15.1404	14.7573	13.5928	13.1087	10.8243
5	21.7091	20.4659	18.7186	18.2152	17.1092	15.6317	15.4842	13.9418	13.6335
6	24.2721	22.4870	20.9717	20.2839	20.2971	17.8602	16.5583	15.4633	14.4789
7	26.2869	24.2644	22.8543	21.6818	20.5974	20.1200	19.0697	17.7064	15.9126
8	28.3030	25.7461	24.3315	23.4497	22.4328	21.1563	20.3632	19.1440	18.1475
9	29.5489	27.9249	26.8101	25.2662	24.2510	22.7821	21.7109	20.2745	19.7147
10	31.7548	29.4709	27.5980	27.0357	25.3011	25.3250	23.4556	22.6180	21.6647

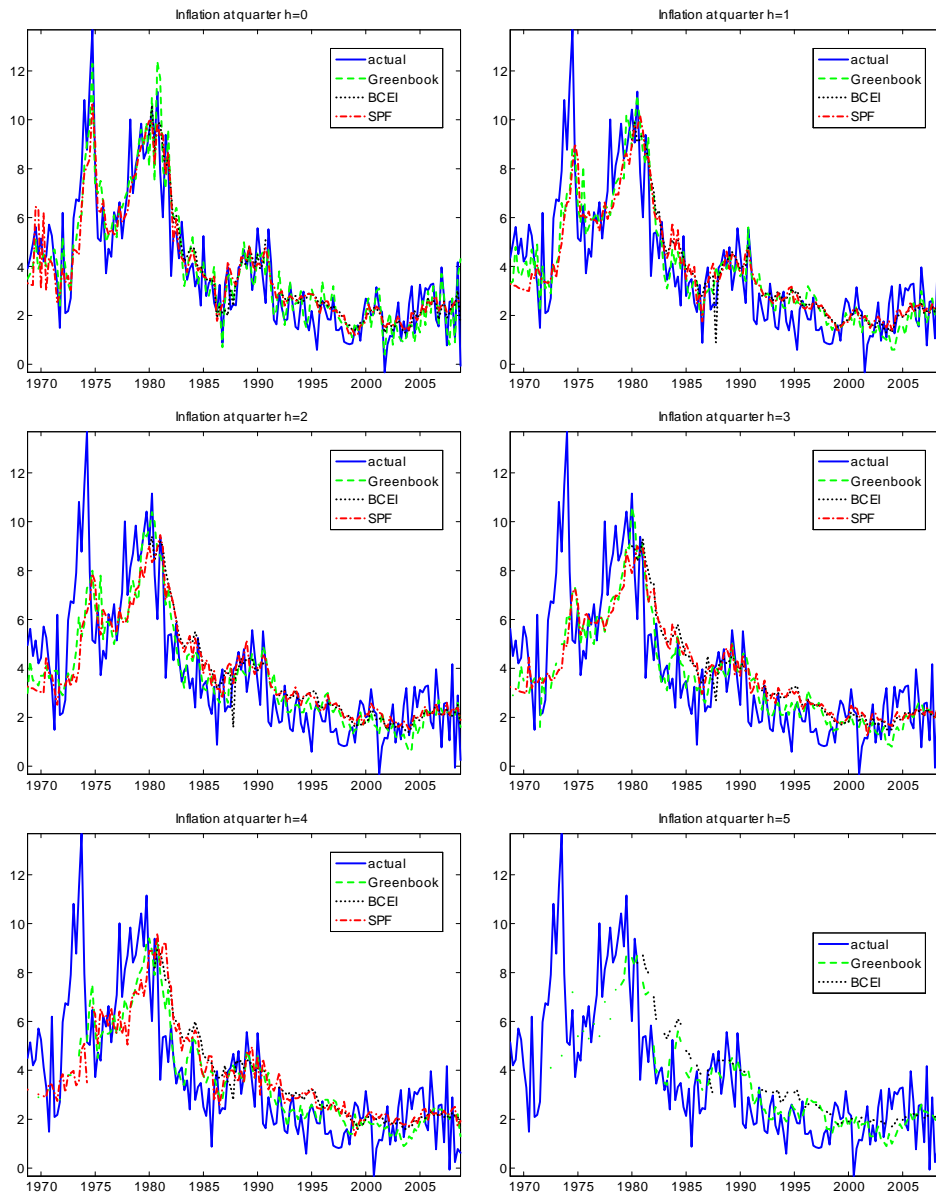
Panel B. 1% Significance Level

ℓ	$\tilde{\mu}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	15.0484	14.0966	12.0133	12.5257	11.2177	10.5051	9.4237	8.5435	8.2030
2	18.5865	15.9096	16.8898	14.1998	14.0067	14.1862	12.5407	12.8689	10.4675
3	22.2589	18.4610	18.1309	16.6749	16.6152	16.6209	15.1820	14.2981	12.7461
4	23.2931	22.6241	20.4215	20.1829	18.5263	18.7161	17.1985	16.8407	15.3503
5	25.9859	24.9017	22.1130	22.3233	22.0671	19.2611	19.5215	17.5694	18.9347
6	28.3701	27.1948	25.1033	24.2014	24.8307	22.2983	20.9119	19.1080	18.0740
7	31.4425	28.9881	27.1075	26.9725	24.3925	24.0897	24.2268	21.7648	20.5248
8	32.4669	29.5149	28.6144	28.1559	26.8484	26.2502	25.2785	23.0942	22.8633
9	33.7114	35.2864	31.3305	29.5492	29.0904	28.2170	25.6922	26.0238	24.8479
10	36.6704	33.1601	32.2650	32.6074	30.0138	31.1782	27.3047	25.9979	26.2313

Note. The table reports the critical values, $\kappa_{\alpha,\ell}$, for several restrictions (ℓ) at $\alpha = 10\%$, 5% and 1% significance levels for $\max_{j \in \{R+m, \dots, T\}} \mathcal{W}_{j,m}$ for the case when parameter estimation error is irrelevant as in Corollary 9 in the paper. Critical values are based on $P = 1000$ and 1000 Monte Carlo simulations; $\tilde{\mu} = m/P$.

4 Additional Data Figures

Figure A.1: Inflation Forecasts



Note. The figure plots Greenbook, BCEI and SPF forecasts of inflation for various forecast horizons h in conjunction with the realized values of inflation, labeled as “actual,” for the corresponding horizon. If a forecast for a specific horizon by the corresponding agency does not exist, it is depicted as a missing value.