Not-for-Publication Appendix to: Forecast Rationality Tests in the Presence of Instabilities, With Applications to Federal Reserve and Survey Forecasts

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This Appendix consists of four sections. Section 1 lists the theoretical results listed in the paper, Section 2 provides their proofs. Section 3 provides additional critical values for the proposed tests and Section 4 plots the complete data used in the empirical application.

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1 List of Results To Be Proved

Mean Value Approximation. *The following mean value approximation holds:*

$$m^{1/2}\widehat{\theta}_j = G^{-1}\left(\frac{T}{m}\right)^{1/2}[I_\ell,\ FB]\left\{\frac{1}{\sqrt{T}}\sum_{t=R}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \frac{1}{\sqrt{T}}\sum_{t=R}^{j-m}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix}\right\} + A_j, \tag{1}$$

where $lim_{T\to\infty} sup_j\ A_j = o_p(1)$.

Proposition 1 (Preliminary Asymptotic Result) *Under Assumptions 1-4 and* $T^{-1/2}\xi_T \to \xi$ *in* $\mathcal{D}_{\mathbb{R}^{(\ell+q)}}[0,1]$*:*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{j}\begin{pmatrix} b_{R,t,j}\cdot I_\ell & 0 \\ 0 & a_{R,t,j}\cdot I_q \end{pmatrix}\begin{pmatrix} f_{t+h} \\ h_t \end{pmatrix} \Rightarrow \int_0^\tau \Omega(s,\tau)^{1/2} d\xi(s),$$

where $\xi(s) = S^{1/2}\mathcal{B}_{\ell+q}(s)$, $\mathcal{B}_{\ell+q}(s)$ *is an* $(\ell+q)\times 1$ *vector of independent standard Brownian motions,* $\mathcal{D}$ *denotes the space of cadlag functions, "*$\Rightarrow$*" denotes weak convergence with respect to the Skorohod metric, and* $\Omega(s,\tau)^{1/2} \equiv \begin{pmatrix} \sigma_f(s)\cdot I_\ell & 0 \\ 0 & \sigma_h(s,\tau)\cdot I_q \end{pmatrix}$.

Proposition 2 (Asymptotic Distribution of $\widehat{\theta}_j$) *Under Assumptions 1-4 and* $T^{-1/2}\xi_T \to \xi$ *in* $\mathcal{D}_{\mathbb{R}^{(\ell+q)}}[0,1]$*:*

$$m^{1/2}\ \widehat{\theta}_j \Rightarrow \int_0^\tau \widetilde{\omega}(s,\tau)\, d\mathcal{B}_{\ell+q}(s) - \int_0^{\tau-\mu}\widetilde{\omega}(s,\tau-\mu)\, d\mathcal{B}_{\ell+q}(s) = \mathcal{B}_{\widetilde{\omega}}(\tau) - \mathcal{B}_{\widetilde{\omega}}(\tau-\mu) \tag{2}$$

$$\underset{d}{=} \int_0^\tau \omega(s,\tau)\, d\mathcal{B}_{\ell+q}(s) \equiv \mathcal{B}_\omega(\tau) = \mathcal{B}_\ell\left(\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\, ds\right), \tag{3}$$

where

$$\widetilde{\omega}(s,\tau) = \mu^{-\frac{1}{2}} G^{-1}[I_\ell, FB]\,\Omega(s,\tau)^{1/2} S^{1/2}, \tag{4}$$

$$\omega(s,\tau) = \mu^{-\frac{1}{2}} G^{-1}[I_\ell, FB]\{\left[\Omega(s,\tau)^{1/2} - \Omega(s,\tau-\mu)^{1/2}\right]\cdot 1(s \le \tau-\mu) \tag{5}$$
$$+\,\Omega(s,\tau)^{1/2}\cdot 1(\tau-\mu < s \le \tau)\} S^{1/2},$$

$\mathcal{B}_{\ell+q}(s)$ *is an* $(\ell+q)\times 1$ *vector of independent standard Brownian motions and* $\underset{d}{=}$ *denotes equality in distribution.*

Proposition 3 (Calculation of $\int_0^{\tau} \omega(s,\tau)\,\omega(s,\tau)'\,ds$)

$$\int_0^{\tau} \omega(s,\tau)\,\omega(s,\tau)'\,ds = \mu^{-1}G^{-1}\Big\{ \left(\int_{\tau-\mu}^{\tau} \sigma_f^2(s)\,ds\right) S_{ff} + \left(\int_{\tau-\mu}^{\tau} \sigma_f(s)\,\sigma_h(s,\tau)\,ds\right) (FBS_{fh} + S_{fh}B'F') + \int_0^{\tau} \left[(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu))^2 \cdot 1(s \le \tau-\mu) + \sigma_h^2(s,\tau) \cdot 1(\tau-\mu \le s \le \tau) \right] ds FBS_{hh}B'F' \Big\} G^{-1},$$

where

(i) $\int_{\tau-\mu}^{\tau} \sigma_f^2(s)\,ds = \mu$ *for both rolling and recursive cases;*

(ii) recursive: let $\widetilde{\pi} \equiv \mu/(\tau-\mu)$;

$\int_{\tau-\mu}^{\tau} \sigma_f(s)\,\sigma_h(s,\tau)\,ds = \mu\left[1 - \widetilde{\pi}^{-1}\ln(1+\widetilde{\pi})\right]$ *and*

$\int_0^{\tau} \left[(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu))^2 \cdot 1(s \le \tau-\mu) + \sigma_h^2(s,\tau) \cdot 1(\tau-\mu \le s < \tau) \right] ds = 2\mu\left[1 - \widetilde{\pi}^{-1}\ln(1+\widetilde{\pi})\right];$

(iii) rolling: let $\pi^{\dagger} \equiv \frac{\mu}{\rho}$;

(a) if $\mu \ge \rho$, *then*

$\int_{\tau-\mu}^{\tau} \sigma_f(s)\,\sigma_h(s,\tau)\,ds = \mu\left(1 - \frac{1}{2\pi^{\dagger}}\right)$ *and*

$\int_0^{\tau} \left[(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu))^2 \cdot 1(s \le \tau-\mu) + \sigma_h^2(s,\tau) \cdot 1(\tau-\mu \le s < \tau) \right] ds = \mu\left(1 - \frac{1}{3\pi^{\dagger}}\right);$

(b) if $\mu < \rho$, *then*

$\int_{\tau-\mu}^{\tau} \sigma_f(s)\,\sigma_h(s,\tau)\,ds = \frac{1}{2}\mu\pi^{\dagger}$ *and*

$\int_0^{\tau} \left[(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu))^2 \cdot 1(s \le \tau-\mu) + \sigma_h^2(s,\tau) \cdot 1(\tau-\mu \le s < \tau) \right] ds = \mu\pi^{\dagger}\left(1 - \frac{1}{3}\pi^{\dagger}\right).$

Proposition 4 (Calculation of $\int_0^{\tau} \widetilde{\omega}(s,\tau)\,\widetilde{\omega}(s,\tau)'\,ds$)

$$\int_0^{\tau} \tilde{\omega}(s,\tau)\,\tilde{\omega}(s,\tau)'\,ds = \mu^{-1}G^{-1}\left\{ \left(\int_0^{\tau} \sigma_f^2(s)\,ds\right) S_{ff} + \left(\int_0^{\tau} \sigma_f(s)\,\sigma_h(s,\tau)\,ds\right) \times \times (FBS_{fh} + S_{fh}B'F') + \left(\int_0^{\tau} \sigma_h^2(s,\tau)\,ds\right) FBS_{hh}B'F' \right\} G^{-1},$$

where

(i) $\int_0^{\tau} \sigma_f^2(s)\,ds = (\tau-\rho)$ *for both rolling and recursive cases;*

(ii) recursive:

$\int_{\tau-\mu}^{\tau} \sigma_h(s,\tau)\,\sigma_f(s)\,ds = (\tau-\rho)\left(1 - \frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)$ *and*

$\int_0^{\tau} \sigma_h^2(s,\tau)\,ds = 2(\tau-\rho)\left(1 - \frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right);$

(iii) rolling:

(a) if $\tau-\rho \ge \rho$, *then*

$\int_0^{\tau} \sigma_f(s)\,\sigma_h(s,\tau)\,ds = \left(\tau - \frac{3}{2}\rho\right)$ *and* $\int_0^{\tau} \sigma_h^2(s)\,ds = \left(\tau - \frac{4}{3}\rho\right);$

(b) if $\tau-\rho < \rho$, *then*

$\int_0^{\tau} \sigma_f(s)\,\sigma_h(s,\tau)\,ds = \frac{1}{2\rho}(\tau-\rho)^2$ *and* $\int_0^{\tau} \sigma_h^2(s)\,ds = \frac{1}{3\rho^2}(\tau-\rho)^2(4\rho-\tau).$

Theorem 5 (Main Proposition) *Under Assumption 1-4,*

$$\mathcal{W}_{j,m} = m\widehat{\theta}_j' \; V_{\theta,\tau}^{-1}\widehat{\theta}_j$$
$$\Rightarrow \left[\mathcal{B}_\ell\left(\int_0^\tau \omega\left(s,\tau\right)\omega\left(s,\tau\right)' ds\right)\right]' V_{\theta,\tau}^{-1}\left[\mathcal{B}_\ell\left(\int_0^\tau \omega\left(s,\tau\right)\omega\left(s,\tau\right)' ds\right)\right], \tag{6}$$

where

$$V_{\theta,\tau} = Avar\left(m^{1/2}\widehat{\theta}_j\right) \tag{7}$$
$$= G^{-1}\left[I_\ell,\; FB\right] Avar\left(\frac{1}{\sqrt{m}}\sum_{t=j-m+1}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix}\right)\left[I_\ell,\; FB\right]' G^{-1}$$
$$= \int_0^\tau \omega\left(s,\tau\right)\omega\left(s,\tau\right)' ds, \tag{8}$$

$j = [\tau T]$, $m = [\mu T]$ *and* $\mathcal{B}_\ell(\cdot)$ *is a standard* ℓ*-dimensional Brownian motion. Let* θ_j *be the true parameter value. We reject the null hypothesis:*

$$H_0 : \theta_j = \theta_0 \; , \; \theta_0 = 0 \text{ for all } j = R+m, ..., T \tag{9}$$

if $max_{j\in\{R+m,...,T\}}\; \mathcal{W}_{j,m} > \kappa_{\alpha,\ell}$*, where* $\kappa_{\alpha,\ell}$ *are the critical values at the* $100\alpha\%$ *significance level that can be simulated for given values of* μ*,* ℓ*,* G*,* F*,* B *and* S*.*

Proposition 6 (Special Case I: Irrelevant Parameter Estimation Error) *Under the condition* $F = 0$*, parameter estimation error becomes irrelevant and* $\int_0^\tau \omega(s,\tau)\omega(s,\tau)' ds$ *becomes* $G^{-1}S_{ff}G^{-1}$ *and* $\int_0^\tau \tilde{\omega}(s,\tau)\tilde{\omega}(s,\tau)' ds = \frac{(\tau-\rho)}{\mu}G^{-1}S_{ff}G^{-1}$ *for all estimation schemes.*

Proposition 7 (Special Case II: Forecast Unbiasedness and Efficiency Tests) *Under the condition:*

$$S_{ff} = -\frac{1}{2}(FBS_{hf} + S_{fh}B'F') = FBS_{hh}B'F', \tag{10}$$

$\int_0^\tau \tilde{\omega}(s,\tau)\tilde{\omega}(s,\tau)' ds$ *in Proposition 4 becomes:*
(i) recursive case: $\frac{\tau-\rho}{\mu}G^{-1}S_{ff}G^{-1}$*;*
(ii) rolling case:
(a) $\frac{1}{\mu}\frac{2\rho}{3}G^{-1}S_{ff}G^{-1}$*, if* $\tau-\rho \geq \rho$*; and*
(b)$\frac{(\tau-\rho)}{\mu}\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)G^{-1}S_{ff}G^{-1}$*, if* $\tau-\rho < \rho$*.*
Furthermore, $\int_0^\tau \omega(s,\tau)\omega(s,\tau)' ds$ *in Proposition 3 becomes* $\lambda G^{-1}S_{ff}G^{-1}$*, where:*
(i') recursive case: $\lambda = 1$*;*
(ii') rolling case: let $\pi^\dagger \equiv \frac{\mu}{\rho}$*; then,*
(a) $\lambda = \frac{2}{3\pi^\dagger}$*, if* $\mu \geq \rho$*; and*
(b) $\lambda = \left(1-\frac{1}{3}\left(\pi^\dagger\right)^2\right)$*, if* $\mu < \rho$*.*

Theorem 8 (Main Proposition in Special Cases) *(a) Under Assumption 1-4 and Condition (10), we have:*

$$\mathcal{W}_{j,m} \Rightarrow \mathcal{W}_{\tau,\mu}, \tag{11}$$

where $\mathcal{W}_{\tau,\mu}$ is: (i) Recursive estimation:

$$\mathcal{W}_{\tau,\mu} = \mu^{-1} \left[\mathcal{B}_\ell (\tau - \rho) - \mathcal{B}_\ell (\tau - \rho - \mu)\right]' \left[\mathcal{B}_\ell (\tau - \rho) - \mathcal{B}_\ell (\tau - \rho - \mu)\right], \tag{12}$$

(ii) Rolling estimation:

$$\begin{aligned}\mathcal{W}_{\tau,\mu} = \mu^{-1} &\left\{ \left(\frac{2}{3\pi^\dagger}\right) \cdot 1 (\mu \geq \rho) + \left(1 - \frac{1}{3}\left(\pi^\dagger\right)^2\right) \cdot 1 (\mu < \rho) \right\}^{-1} \times \\ &\Bigg[\left\{ \mathcal{B}_\ell \left((\tau-\rho) \left(1 - \frac{(\tau-\rho)^2}{3\rho^2}\right)\right) - \mathcal{B}_\ell \left((\tau-\mu-\rho) \left(1 - \frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right) \right\} \cdot 1 (\mu + \rho \leq \tau < 2\rho) \\ &+ \left\{ \mathcal{B}_\ell \left(\frac{2}{3}\rho\right) - \mathcal{B}_\ell \left((\tau-\mu-\rho) \left(1 - \frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right) \right\} \cdot 1 (2\rho < \tau \leq 2\rho + \mu) \Bigg]' \\ &\times \Bigg[\left\{ \mathcal{B}_\ell \left((\tau-\rho) \left(1 - \frac{(\tau-\rho)^2}{3\rho^2}\right)\right) - \mathcal{B}_\ell \left((\tau-\mu-\rho) \left(1 - \frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right) \right\} \cdot 1 (\mu + \rho \leq \tau < 2\rho) \\ &+ \left\{ \mathcal{B}_\ell \left(\frac{2}{3}\rho\right) - \mathcal{B}_\ell \left((\tau-\mu-\rho) \left(1 - \frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right) \right\} \cdot 1 (2\rho < \tau \leq 2\rho + \mu) \Bigg].\end{aligned} \tag{13}$$

(b) Furthermore, under Assumptions 1-4 and condition $F = 0$, eq. (11) holds with

$$\mathcal{W}_{\tau,\mu} = \mu^{-1} \left[\mathcal{B}_\ell (\tau - \rho) - \mathcal{B}_\ell (\tau - \rho - \mu)\right]' \left[\mathcal{B}_\ell (\tau - \rho) - \mathcal{B}_\ell (\tau - \rho - \mu)\right]. \tag{14}$$

We reject the null hypothesis:

$$H_0 : \theta_j = \theta_0,\ \theta_0 = 0 \text{ for all } j = R + m, ..., T \tag{15}$$

if $\max_{j \in \{R+m,...,T\}} W_{j,m} > \kappa_{\alpha,\ell}$, where $\kappa_{\alpha,\ell}$ are the critical values at the $100\alpha\%$ significance level and are reported for $\alpha = 0.05$ in Table 1, Panel A for eq. (12) and (14) for various values of $\mu = m/T$ and number of restrictions, ℓ; and in Table 1, Panel B for eq. (13) for various combinations of μ, ρ, ℓ

2 Proofs

Proof of Equation (1). Note that

$$\widehat{\theta}_j = \left(m^{-1}\sum_{t=j-m+1}^{j}\widehat{g}_t\widehat{g}_t'\right)^{-1}\left(m^{-1}\sum_{t=j-m+1}^{j}\widehat{g}_t v_{t+h}\left(\widehat{\gamma}_{t,R}\right)\right) \tag{16}$$
$$= \left(m^{-1}\sum_{t=j-m+1}^{j}\widehat{g}_t\widehat{g}_t'\right)^{-1}\left(m^{-1}\sum_{t=j-m+1}^{j} f_{t+h}\left(\widehat{\gamma}_{t,R}\right)\right).$$

From a mean value expansion of $f_{t+h}\left(\widehat{\gamma}_{t,R}\right)$ around γ^* we have:

$$f_{t+h}\left(\widehat{\gamma}_{t,R}\right) = f_{t+h} + f_{t+h,\gamma}\left(\widehat{\gamma}_{t,R} - \gamma^*\right) + w_{t+h},$$

where w_{t+h} is the remainder. Furthermore, by Assumption 1(i),

$$m^{-1/2}\sum_{t=j-m+1}^{j} f_{t+h}\left(\widehat{\gamma}_{t,R}\right) = m^{-1/2}\sum_{t=j-m+1}^{j} f_{t+h} + m^{-1/2}\sum_{t=j-m+1}^{j} f_{t+h,\gamma}B_tH_t \tag{17}$$
$$+ m^{-1/2}\sum_{t=j-m+1}^{j} w_{t+h},$$

As in the proof of equation (4.1) in West (1996), note that

$$m^{-1/2}\sum_{t=j-m+1}^{j} f_{t+h,\gamma}B_tH_t = m^{-1/2}FB\sum_{t=j-m+1}^{j} H_t + \widetilde{A}_j, \text{ where} \tag{18}$$

$$\widetilde{A}_j \equiv m^{-1/2}\sum_{t=j-m+1}^{j}\left(f_{t+h,\gamma} - F\right)BH_t + m^{-1/2}F\sum_{t=j-m+1}^{j}\left(B_t - B\right)H_t$$
$$+ m^{-1/2}\sum_{t=j-m+1}^{j}\left(f_{t+h,\gamma} - F\right)\left(B_t - B\right)H_t.$$

Assumption 2 implies that the last three terms in $\widetilde{A}_j$ are $o_p(1)$.

Therefore, by equations (16), (17) and (18), we have:

$$m^{1/2}\widehat{\theta}_j = \left(m^{-1}\sum_{t=j-m+1}^{j}\widehat{g}_t\widehat{g}_t'\right)^{-1}\left(m^{-1/2}\sum_{t=j-m+1}^{j} f_{t+h}\left(\widehat{\gamma}_{t,R}\right)\right)$$
$$= \left(m^{-1}\sum_{t=j-m+1}^{j}\widehat{g}_t\widehat{g}_t'\right)^{-1}\left(m^{-1/2}\sum_{t=j-m+1}^{j} f_{t+h} + m^{-1/2}FB\sum_{t=j-m+1}^{j} H_t + \widetilde{A}_j + m^{-1/2}\sum_{t=j-m+1}^{j} w_{t+h}\right)$$
$$= \left(m^{-1}\sum_{t=j-m+1}^{j}\widehat{g}_t\widehat{g}_t'\right)^{-1}\left[I_\ell, FB\right]\left\{m^{-1/2}\sum_{t=j-m+1}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix}\right\} + A_j^{\dagger},$$

where

$$A_j^{\dagger} \equiv \left(m^{-1}\sum_{t=j-m+1}^{j}\widehat{g}_t\widehat{g}_t'\right)^{-1}\left[\widetilde{A}_j + m^{-1/2}\sum_{t=j-m+1}^{j} w_{t+h}\right].$$

Thus,

$$\begin{aligned} m^{1/2}\widehat{\theta}_j &= \left(m^{-1}\sum_{t=j-m+1}^{j}\widehat{g}_t\widehat{g}_t'\right)^{-1}[I_\ell, FB]\left(\frac{T}{m}\right)^{1/2}\left\{\frac{1}{T^{1/2}}\sum_{t=R}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \frac{1}{T^{1/2}}\sum_{t=R}^{j-m}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix}\right\} + A_j^{\dagger} \\ &= G^{-1}[I_\ell, FB]\left(\frac{T}{m}\right)^{1/2}\frac{1}{T^{1/2}}\left\{\sum_{t=R}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \sum_{t=R}^{j-m}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix}\right\} + A_j, \text{ where} \end{aligned}$$

$$A_j = A_j^{\dagger} + \left\{\left(m^{-1}\sum_{t=j-m+1}^{j}\widehat{g}_t\widehat{g}_t'\right)^{-1} - G^{-1}\right\}[I_\ell, FB]\left(\tfrac{T}{m}\right)^{1/2}\left\{\tfrac{1}{T^{1/2}}\sum_{t=R}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \tfrac{1}{T^{1/2}}\sum_{t=R}^{j-m}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix}\right\}.$$

By Assumption 2(v) and arguments similar to West (1996, proof of equation 4.1) and West and McCracken (1998, Lemma 4.1), $lim_{T\to\infty}sup_j\left|m^{-1/2}\sum_{t=j-m+1}^{j} w_{t+h}\right| = o_p(1)$. In addition, from arguments similar to those in Lemma 4.3 in West and McCracken (1998), Assumption 2 and consistency of $\widehat{\gamma}_{t,R}$, $lim_{T\to\infty}sup_j\left(\left(m^{-1}\sum_{t=j-m+1}^{j}\widehat{g}_t\widehat{g}_t'\right)^{-1} - G^{-1}\right) = 0$**.** Therefore, Assumptions 1 and 2 ensure that $lim_{T\to\infty}sup_j\ A_j = o_p(1)$. ■

Proof of Proposition 1. By Hansen (1992), under Assumptions 1-4 and $T^{-1/2}\xi_T \to \xi$ in $\mathcal{D}_{\mathbb{R}^{(\ell+q)}}[0,1]$ then

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{j}\begin{pmatrix} b_{R,t,j}\cdot I_\ell & 0 \\ 0 & a_{R,t,j}\cdot I_q \end{pmatrix}\begin{pmatrix} f_{t+h} \\ h_t \end{pmatrix} - C_T^*(\tau) \Rightarrow \int_0^\tau \begin{pmatrix} \sigma_f(s)\cdot I_\ell & 0 \\ 0 & \sigma_h(s,\tau)\cdot I_q \end{pmatrix} d\xi(s),$$

where $\xi(s) = S^{1/2}\mathcal{B}_{l+q}(s)$, $z_t = \sum_{k=1}^{\infty} E_t\left(\begin{bmatrix} f_{t+h+k} & h_{t+k} \end{bmatrix}'\right)$ and

$$\begin{aligned} C_T^*(\tau) = \Bigg\{ & T^{-1/2}\sum_{t=1}^{[\tau T]}\left[\begin{pmatrix} b_{R,t,j}\cdot I_\ell & 0 \\ 0 & a_{R,t,j}\cdot I_q \end{pmatrix} - \begin{pmatrix} b_{R,t-1,j}\cdot I_\ell & 0 \\ 0 & a_{R,t-1,j}\cdot I_q \end{pmatrix}\right] z_t \\ & - T^{-1/2}\begin{pmatrix} b_{R,j,j}\cdot I_\ell & 0 \\ 0 & a_{R,j,j}\cdot I_q \end{pmatrix} z_{j+1}\Bigg\}. \end{aligned}$$

The proof follows from the fact that $\sup_\tau C_T^*(\tau) = o_p(1)$, using the same reasoning as in Cavaliere (2004, Proof of Theorem 4), and the fact that the variances $\sigma_f(s), \sigma_h(s,\tau)$ are square integrable and bounded. ■

Proof of Proposition 2. It follows directly from Proposition 1 and Assumption 2 that

$$T^{-1/2}\sum_{t=R}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} \Rightarrow \int_0^{\tau}\Omega\left(s,\tau\right)^{1/2}S^{1/2}d\mathcal{B}_{l+q}\left(s\right).$$

Thus,

$$\begin{aligned}
&m^{1/2}\,\widehat{\theta}_j = G^{-1}\left(\frac{T}{m}\right)^{1/2}\left[I_\ell,FB\right]\left(\frac{1}{\sqrt{T}}\sum_{t=R}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \frac{1}{\sqrt{T}}\sum_{t=R}^{j-m}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix}\right) + A_j \Rightarrow \\
&\mu^{-1/2}G^{-1}\left[I_\ell,FB\right]\left(\int_0^{\tau}\Omega\left(s,\tau\right)^{1/2}S^{1/2}d\mathcal{B}_{l+q}\left(s\right) - \int_0^{\tau-\mu}\Omega\left(s,\tau-\mu\right)^{1/2}S^{1/2}d\mathcal{B}_{l+q}\left(s\right)\right) \qquad (19)\\
&= \mu^{-1/2}G^{-1}\left[I_\ell,FB\right]\begin{pmatrix} \int_0^{\tau-\mu}\left[\Omega\left(s,\tau\right)^{1/2} - \Omega\left(s,\tau-\mu\right)^{1/2}\right]S^{1/2}d\mathcal{B}_{l+q}\left(s\right) \\ +\int_{\tau-\mu}^{\tau}\Omega\left(s,\tau\right)^{1/2}S^{1/2}d\mathcal{B}_{l+q}\left(s\right) \end{pmatrix}\\
&= \mu^{-1/2}G^{-1}\left[I_\ell,FB\right]\int_0^{\tau}\begin{pmatrix} \left[\Omega\left(s,\tau\right)^{1/2} - \Omega\left(s,\tau-\mu\right)^{1/2}\right]\cdot 1\left(s\leq\tau-\mu\right) \\ +\Omega\left(s,\tau\right)^{1/2}\cdot 1\left(\tau-\mu<s\leq\tau\right) \end{pmatrix}S^{1/2}d\mathcal{B}_{l+q}\left(s\right)\\
&= \int_0^{\tau}\omega\left(s,\tau\right)d\mathcal{B}_{l+q}\left(s\right) = \mathcal{B}_l\left(\int_0^{\tau}\omega\left(s,\tau\right)\omega\left(s,\tau\right)'ds\right),
\end{aligned}$$

where $\omega\left(s,\tau\right)$, $\widetilde{\omega}\left(s,\tau\right)$ are defined in Proposition 2. The second line follows from Assumptions 2 and 3 as well as Proposition 1; the last equality follows from Lemma 2 in Cavaliere (2004). ■

Proof of Proposition 3. Note that:

$$\begin{aligned}
&\omega\left(s,\tau\right) = \mu^{-\frac{1}{2}}G^{-1}\left[I_\ell,FB\right]\begin{bmatrix} \left(\Omega\left(s,\tau\right)^{1/2} - \Omega\left(s,\tau-\mu\right)\right)\cdot 1\left(s\leq\tau-\mu\right) \\ +\Omega\left(s,\tau\right)^{1/2}\cdot 1\left(\tau-\mu<s\leq\tau\right) \end{bmatrix}S^{1/2}\\
&= \mu^{-\frac{1}{2}}G^{-1}\left[I_\ell,FB\right]\begin{bmatrix} \sigma_f\left(s\right)\cdot 1\left(\tau-\mu\leq s<\tau\right)\cdot I_\ell & 0 \\ 0 & \begin{matrix}\left(\sigma_h\left(s,\tau\right)-\sigma_h\left(s,\tau-\mu\right)\right)\cdot 1\left(s\leq\tau-\mu\right) \\ +\sigma_h\left(s,\tau\right)\cdot 1\left(\tau-\mu\leq s\leq\tau\right)\cdot I_q\end{matrix} \end{bmatrix}S^{1/2}
\end{aligned}$$

(*i*) $\int_{\tau-\mu}^{\tau}\sigma_f^2\left(s\right)ds = \int_{\tau-\mu}^{\tau}\left(1\left(s\geq\rho\right)\right)^2 ds = \int_{\tau-\mu}^{\tau}ds = \mu;$

(ii) Recursive case:[1] Let $\widetilde{\pi} \equiv \mu / (\tau - \mu)$.

$$\begin{aligned}
\int_{\tau-\mu}^{\tau} \sigma_f(s)\,\sigma_h(s,\tau)\,ds &= \int_{\tau-\mu}^{\tau} 1(s \geq \rho) \cdot ([\ln(\tau) - \ln(\rho)] \cdot 1(s \leq \rho) + [\ln(\tau) - \ln(s)] \cdot 1(s > \rho))ds \\
&= \int_{\tau-\mu}^{\tau} [\ln(\tau) - \ln(s)]\,ds = \int_{\tau-\mu}^{\tau} \ln(\tau)\,ds - \int_{\tau-\mu}^{\tau} \ln(s)\,ds \\
&= \ln(\tau)(\tau - \tau + \mu) - (\ln(\tau)\tau - \tau) + (\ln(\tau-\mu)(\tau-\mu) - (\tau-\mu)) \\
&= -\ln(\tau)(\tau-\mu) + \tau + \ln(\tau-\mu)(\tau-\mu) - \tau + \mu \\
&= \mu - (\tau-\mu)\ln\left(\frac{\tau}{\tau-\mu}\right) = \mu\left[1 - \widetilde{\pi}^{-1}\ln(1+\widetilde{\pi})\right],
\end{aligned}$$

Furthermore,

$$\int_0^{\tau-\mu} (\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu))^2\,ds = \int_0^{\tau-\mu} [\ln(\tau) - \ln(\tau-\mu)]^2\,ds = (\tau-\mu)\left[\ln\left(\frac{\tau}{\tau-\mu}\right)\right]^2.$$

$$\begin{aligned}
\int_{\tau-\mu}^{\tau} \sigma_h^2(s,\tau)\,ds &= \int_{\tau-\mu}^{\tau} ([\ln(\tau) - \ln(\rho)] \cdot 1(s < \rho) + [\ln(\tau) - \ln(s)] \cdot 1(s \geq \rho))^2 ds \\
&= \int_{\tau-\mu}^{\tau} [\ln(\tau) - \ln(s)]^2\,ds = \int_{\tau-\mu}^{\tau} \left(\ln(\tau)^2 - 2\ln(\tau)\ln(s) + \ln(s)^2\right) ds \\
&= \ln(\tau)^2\mu - 2\ln(\tau)[\ln(\tau)\tau - \tau - \ln(\tau-\mu)(\tau-\mu) + \tau - \mu] + \\
&+ \ln(\tau)^2\tau - 2\tau\ln(\tau) + 2\tau - \ln(\tau-\mu)^2(\tau-\mu) + 2(\tau-\mu)\ln(\tau-\mu) - 2(\tau-\mu) \\
&= \ln(\tau)^2\mu - \ln(\tau)^2\tau + 2\ln(\tau)\ln(\tau-\mu)(\tau-\mu) \\
&+ 2\ln(\tau)\mu - 2\tau\ln(\tau) - \ln(\tau-\mu)^2(\tau-\mu) + 2(\tau-\mu)\ln(\tau-\mu) + 2\mu \\
&= 2\mu + \ln(\tau)^2(\mu-\tau) - \ln(\tau-\mu)^2(\tau-\mu) + 2\ln(\tau)\ln(\tau-\mu)(\tau-\mu) \\
&+ 2(\tau-\mu)\ln(\tau-\mu) + 2\ln(\tau)(\mu-\tau) \\
&= 2\mu + \ln(\tau)^2(\mu-\tau) - \ln(\tau-\mu)^2(\tau-\mu) \\
&+ 2\ln(\tau)\ln(\tau-\mu)(\tau-\mu) + 2(\tau-\mu)\ln\left(\frac{\tau-\mu}{\tau}\right) \\
&= 2\mu - 2(\tau-\mu)\ln\left(\frac{\tau}{\tau-\mu}\right) - (\tau-\mu)\left[\ln\left(\frac{\tau}{\tau-\mu}\right)\right]^2
\end{aligned}$$

[1] Note $\int ln(x)dx = ln(x)x - x + c$; $\int ln(x)^2 dx = xln(x)^2 - 2xln(x) + 2x + c$

$$\int_0^{\tau} \left[\left(\sigma_h\left(s,\tau\right) - \sigma_h\left(s,\tau-\mu\right) \right)^2 \cdot 1\left(s \le \tau-\mu\right) + \sigma_h^2\left(s,\tau\right) \cdot 1\left(\tau-\mu < s < \tau\right) \right] ds$$
$$= \int_0^{\tau-\mu} \left(\sigma_h\left(s,\tau\right) - \sigma_h\left(s,\tau-\mu\right) \right)^2 ds + \int_{\tau-\mu}^{\tau} \sigma_h^2\left(s,\tau\right) ds$$
$$= \left(\tau-\mu\right) \left[\ln\left(\frac{\tau}{\tau-\mu} \right) \right]^2 + 2\mu - 2\left(\tau-\mu\right) \ln\left(\frac{\tau}{\tau-\mu} \right) - \left(\tau-\mu\right) \left[\ln\left(\frac{\tau}{\tau-\mu} \right) \right]^2$$
$$= 2\mu - 2\left(\tau-\mu\right) \ln\left(\frac{\tau}{\tau-\mu} \right) = 2\mu \left[1 - \widetilde{\pi}^{-1} \ln\left(1+\widetilde{\pi}\right) \right]$$

(iii) Rolling case: Let $\pi^{\dagger} \equiv \frac{\mu}{\rho}$.

In the rolling estimation scheme there are two possible cases. Case (a) occurs when $\tau - \rho \ge \rho$, while (b) when $\tau - \rho < \rho$. We consider the calculation of the respective integrals in these two cases. We show that the covariance is the same in both cases, no matter whether $\mu \ge \rho$ or $\mu < \rho$.

Case (a): $\tau - \rho \ge \rho$.

This allows for two sub-cases: (i) $\mu \ge \rho \Leftrightarrow \tau - \rho \ge \tau - \mu \ge \rho$ and (ii) $\mu < \rho \Leftrightarrow \tau - \mu > \tau - \rho \ge \rho$ (recall that $\tau \ge \rho + \mu$).

In case (i),

$$\int_{\tau-\mu}^{\tau} \sigma_h\left(s,\tau\right) \sigma_f\left(s\right) ds = \int_{\tau-\mu}^{\tau} 1\left(s \ge \rho\right) \cdot \frac{1}{\rho} \left[\begin{array}{c} s \cdot 1(s \le \rho) + \rho \cdot 1(\rho < s \le \tau-\rho) \\ +(\tau - s) \cdot 1(s > \tau - \rho) \end{array} \right] ds =$$
$$= \int_{\tau-\mu}^{\tau} \frac{1}{\rho} [\rho \cdot 1(\rho \le s \le \tau-\rho) + (\tau - s) \cdot 1(s > \tau-\rho)] ds =$$
$$= (\mu - \rho) + \frac{1}{2}\rho = \mu - \frac{1}{2}\rho = \mu \left(1 - \frac{1}{2}\frac{\rho}{\mu} \right) = \mu \left(1 - \frac{1}{2\pi^{\dagger}} \right);$$

whereas in case (ii),

$$\int_{\tau-\mu}^{\tau} \sigma_h\left(s,\tau\right) \sigma_f\left(s\right) ds = \int_{\tau-\mu}^{\tau} 1\left(s \ge \rho\right) \cdot \frac{1}{\rho} \left[\begin{array}{c} s \cdot 1(s \le \rho) + \rho \cdot 1(\rho < s \le \tau-\rho) \\ +(\tau - s) \cdot 1(s > \tau - \rho) \end{array} \right] ds =$$
$$= \int_{\tau-\mu}^{\tau} \frac{1}{\rho} [(\tau - s) \cdot 1(s > \tau-\rho)] ds = \int_{\tau-\mu}^{\tau} \frac{1}{\rho} (\tau - s) ds = \frac{1}{2}\frac{\mu^2}{\rho} = \frac{1}{2}\mu\pi^{\dagger}.$$

Furthermore,

$$\int_0^{\tau-\mu}\left(\sigma_h\left(s,\tau\right)-\sigma_h\left(s,\tau-\mu\right)\right)^2 ds =$$

$$\left(\frac{1}{\rho}\right)^2\int_0^{\tau-\mu}\left(\begin{array}{c}[s\cdot 1(s\leq\rho)+\rho\cdot 1(\rho<s\leq\tau-\rho)+(\tau-s)\cdot 1(\tau-\rho<s\leq\tau)]-\\ [s\cdot 1(s\leq\rho)+\rho\cdot 1(\rho<s\leq\tau-\mu-\rho)+\\ (\tau-\mu-s)\cdot 1(\tau-\mu-\rho<s\leq\tau-\mu)]\end{array}\right)^2 ds =$$

$$\left(\frac{1}{\rho}\right)^2\int_0^{\tau-\mu}\left(\begin{array}{c}\rho\cdot 1(\tau-\mu-\rho\leq s\leq\tau-\rho)+(\tau-s)\cdot 1(\tau-\rho<s\leq\tau)\\ -(\tau-\mu-s)\cdot 1(\tau-\mu-\rho<s\leq\tau-\mu)\end{array}\right)^2 ds$$

The expression above simplifies:

(i) $\mu\geq\rho\Leftrightarrow\tau-\rho>\tau-\mu\geq\rho$

$$\int_0^{\tau-\mu}\left(\sigma_h\left(s,\tau\right)-\sigma_h\left(s,\tau-\mu\right)\right)^2 ds=\left(\frac{1}{\rho}\right)^2\int_{\tau-\mu-\rho}^{\tau-\mu}\left(\rho-(\tau-\mu-s)\right)^2 ds=\frac{1}{3}\rho.$$

In addition,

$$\int_{\tau-\mu}^{\tau}\sigma_h^2\left(s,\tau\right)ds=\left(\frac{1}{\rho}\right)^2\int_{\tau-\mu}^{\tau}[s\cdot 1(s\leq\rho)+\rho\cdot 1(\rho<s\leq\tau-\rho)+(\tau-s)\cdot 1(s>\tau-\rho)]^2 ds=$$
$$=\left(\frac{1}{\rho}\right)^2\left(\int_{\tau-\mu}^{\tau-\rho}\rho^2 ds+\int_{\tau-\rho}^{\tau}(\tau-s)^2 ds\right)=(\mu-\rho)+\frac{1}{3}\rho=\mu-\frac{2}{3}\rho.$$

Thus,

$$\int_0^{\tau-\mu}\left(\sigma_h\left(s,\tau\right)-\sigma_h\left(s,\tau-\mu\right)\right)^2 ds+\int_{\tau-\mu}^{\tau}\sigma_h^2\left(s\right)ds=\mu-\frac{1}{3}\rho=\mu\left(1-\frac{1}{3\pi^\dagger}\right).$$

(ii) $\mu<\rho\Leftrightarrow\tau-\mu>\tau-\rho\geq\rho$

$$\int_0^{\tau-\mu}\left(\sigma_h\left(s,\tau\right)-\sigma_h\left(s,\tau-\mu\right)\right)^2 ds =$$

$$\left(\frac{1}{\rho}\right)^2\int_0^{\tau-\mu}\left(\begin{array}{c}(\rho-(\tau-\mu-s))\cdot 1(\tau-\mu-\rho\leq s\leq\tau-\rho)+(\tau-s)\cdot 1(\tau-\rho<s<\tau)\\ -(\tau-\mu-s)\cdot 1(\tau-\mu>s>\tau-\rho)\end{array}\right)^2 ds$$

$$\left(\frac{1}{\rho}\right)^2\int_0^{\tau-\mu}\left[(\rho-(\tau-\mu-s))\cdot 1(\tau-\mu-\rho\leq s\leq\tau-\rho)+\mu\cdot 1(\tau-\mu>s>\tau-\rho)\right]^2 ds$$

$$\left(\frac{1}{\rho}\right)^2\int_{\tau-\mu-\rho}^{\tau-\rho}\left[\rho-(\tau-\mu-s)\right]^2 ds+\left(\frac{\mu}{\rho}\right)^2\int_{\tau-\rho}^{\tau-\mu}ds=-\frac{1}{3}\frac{\mu^2}{\rho^2}\left(2\mu-3\rho\right)=\frac{\mu^2}{\rho}-\frac{2}{3}\frac{\mu^3}{\rho^2}$$

In addition,

$$\int_{\tau-\mu}^{\tau}\sigma_h^2(s,\tau)\,ds=\left(\frac{1}{\rho}\right)^2\int_{\tau-\mu}^{\tau}[s\cdot 1(s\leq\rho)+\rho\cdot 1(\rho<s\leq\tau-\rho)+(\tau-s)\cdot 1(s>\tau-\rho)]^2ds=$$
$$=\left(\frac{1}{\rho}\right)^2\int_{\tau-\mu}^{\tau}(\tau-s)^2ds=\frac{1}{3}\frac{\mu^3}{\rho^2}$$

Thus,

$$\int_0^{\tau-\mu}\left(\sigma_h(s,\tau)-\sigma_h(s,\tau-\mu)\right)^2ds+\int_{\tau-\mu}^{\tau}\sigma_h^2(s)\,ds=\frac{\mu^2}{\rho}-\frac{2}{3}\frac{\mu^3}{\rho^2}+\frac{1}{3}\frac{\mu^3}{\rho^2}=\mu\pi^\dagger\left(1-\frac{1}{3\pi^\dagger}\right).$$

Case (b): $\tau-\rho<\rho$.

Note that, since $\tau\geq\rho+\mu$, in this case the only possible subcase is $\mu<\rho$. Thus,

$$\int_{\tau-\mu}^{\tau}\sigma_f(s)\,\sigma_h(s,\tau)\,ds=\frac{1}{\rho}\int_{\tau-\mu}^{\tau}1(s\geq\rho)\cdot\left[\begin{array}{c}s\cdot 1(s<\tau-\rho)+(\tau-\rho)\cdot 1(\tau-\rho\leq s\leq\rho)\\+(\tau-s)\cdot 1(s>\rho)\end{array}\right]ds=$$
$$=\frac{1}{\rho}\int_{\tau-\mu}^{\tau}(\tau-s)\cdot 1(s>\rho)ds=\frac{1}{\rho}\int_{\tau-\mu}^{\tau}(\tau-s)ds=\frac{1}{2}\frac{\mu^2}{\rho}=\frac{1}{2}\mu\pi^\dagger$$

Furthermore,

$$\int_0^{\tau-\mu}\left(\sigma_h(s,\tau)-\sigma_h(s,\tau-\mu)\right)^2ds$$
$$=\left(\frac{1}{\rho}\right)^2\int_0^{\tau-\mu}\left(\begin{array}{c}s\cdot 1(s<\tau-\rho)+(\tau-\rho)\cdot 1(\tau-\rho\leq s\leq\rho)+(\tau-s)\cdot 1(s>\rho)-\\\left(\begin{array}{c}s\cdot 1(s<\tau-\mu-\rho)+(\tau-\mu-\rho)\cdot 1(\tau-\mu-\rho\leq s\leq\rho)\\+(\tau-\mu-s)\cdot 1(s>\rho)\end{array}\right)\end{array}\right)^2ds$$
$$=\left(\frac{1}{\rho}\right)^2\int_0^{\tau-\mu}\left(\begin{array}{c}(s-(\tau-\mu-\rho))\cdot 1(\tau-\mu-\rho<s<\tau-\rho)+\\((\tau-\rho)-(\tau-\mu-\rho))\cdot 1(\tau-\rho\leq s\leq\rho)+(\tau-s-(\tau-\mu-s))\cdot 1(s>\rho)\end{array}\right)^2ds$$
$$=\left(\frac{1}{\rho}\right)^2\left(\int_{\tau-\mu-\rho}^{\tau-\rho}(s-(\tau-\mu-\rho))^2ds+\int_{\tau-\rho}^{\rho}\mu^2ds+\int_{\rho}^{\tau-\mu}\mu^2ds\right)$$
$$=-\frac{1}{\rho^2}\left(\mu^2(\mu-\tau+\rho)-\frac{1}{3}\mu^3+\mu^2(\tau-2\rho)\right)=\left(\frac{\mu}{\rho}\right)^2\left(\frac{1}{3}\mu-(\mu-\rho)\right)$$

In addition, $\int_{\tau-\mu}^{\tau}\sigma_h^2(s,\tau)\,ds=\frac{1}{\rho^2}\int_{\tau-\mu}^{\tau}(\tau-s)^2ds=\frac{1}{3}\frac{\mu^3}{\rho^2}$. Furthermore,

$$\int_0^{\tau-\mu}\left(\sigma_h(s,\tau)-\sigma_h(s,\tau-\mu)\right)^2ds+\int_{\tau-\mu}^{\tau}\sigma_h^2(s)\,ds=\left(\frac{1}{\rho}\right)^2\left(\frac{1}{3}\mu^3-\mu^2(\mu-\rho)\right)+\frac{1}{3}\frac{\mu^3}{\rho^2}=\mu\pi^\dagger\left(1-\frac{1}{3\pi^\dagger}\right).$$

■

Proof of Proposition 4. From Proposition 2 (in particular, equation 19),

$$m^{1/2}\widehat{\theta}_j \Rightarrow \mu^{-\frac{1}{2}}G^{-1}\left[I_\ell, FB\right]\left(\int_0^{\tau}\Omega\left(s,\tau\right)^{1/2}S^{1/2}d\mathcal{B}_{l+q}\left(s\right)-\int_0^{\tau-\mu}\Omega\left(s,\tau-\mu\right)^{1/2}S^{1/2}d\mathcal{B}_{l+q}\left(s\right)\right).$$

By arguments similar to those in Proposition 2

$$m^{1/2}\widehat{\theta}_j \Rightarrow \mathcal{B}\left(\int_0^{\tau}\tilde{\omega}\left(s,\tau\right)\tilde{\omega}\left(s,\tau\right)'ds\right)-\mathcal{B}\left(\int_0^{\tau-\mu}\tilde{\omega}\left(s,\tau\right)\tilde{\omega}\left(s,\tau\right)'ds\right)$$

where $\tilde{\omega}\left(s,\tau\right)=\mu^{-\frac{1}{2}}G^{-1}\left[I_\ell, FB\right]\Omega\left(s,\tau\right)^{1/2}S^{1/2}$.

Furthermore,

$$\int_0^{\tau}\tilde{\omega}\left(s,\tau\right)\tilde{\omega}\left(s,\tau\right)'ds=\mu^{-1}G^{-1}\left[I_\ell, FB\right]\int_0^{\tau}\Omega\left(s,\tau\right)^{1/2}S\Omega'\left(s,\tau\right)^{1/2}ds\left[I_\ell, FB\right]'G^{-1}=$$

$$\mu^{-1}G^{-1}\left[I_\ell, FB\right]\begin{bmatrix}\int_0^{\tau}\sigma_f^2\left(s\right)dsS_{ff} & \int_0^{\tau}\sigma_f\left(s\right)\sigma_h\left(s,\tau\right)dsS_{fh}\\ \int_0^{\tau}\sigma_f\left(s\right)\sigma_h\left(s,\tau\right)dsS_{fh}' & \int_0^{\tau}\sigma_h^2\left(s,\tau\right)dsS_{hh}\end{bmatrix}\begin{bmatrix}I_\ell\\ B'F'\end{bmatrix}G^{-1}.$$

(i) $\int_0^{\tau}\sigma_f^2\left(s\right)ds=\left(\tau-\rho\right)$;

(ii) Recursive case:

$$\begin{aligned}\int_0^{\tau}\sigma_f\left(s\right)\sigma_h\left(s,\tau\right)ds &= \int_0^{\tau}1\left(s\geq\rho\right)\cdot\left(\left[\ln\left(\tau\right)-\ln\left(\rho\right)\right]\cdot 1\left(s\leq\rho\right)+\left[\ln\left(\tau\right)-\ln\left(s\right)\right]\cdot 1\left(s>\rho\right)\right)ds\\ &= \int_{\rho}^{\tau}\left[\ln\left(\tau\right)-\ln\left(s\right)\right]ds=\int_{\rho}^{\tau}\ln\left(\tau\right)ds-\int_{\rho}^{\tau}\ln\left(s\right)ds=\\ &= \ln\left(\tau\right)\left(\tau-\rho\right)-\left(\ln(\tau)\tau-\tau\right)+\left(\ln(\rho)\rho-\rho\right)=\\ &= \left(\tau-\rho\right)-\rho\ln\left(\frac{\tau}{\rho}\right)=\left(\tau-\rho\right)\left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)\end{aligned}$$

$$\begin{aligned}
\int_0^\tau \sigma_h^2(s,\tau)\,ds &= \int_0^\tau ([\ln(\tau)-\ln(\rho)]\cdot 1(s<\rho)+[\ln(\tau)-\ln(s)]\cdot 1(s\geq\rho))^2 ds \\
&= \int_0^\rho [\ln(\tau)-\ln(\rho)]^2\,ds + \int_\rho^\tau [\ln(\tau)-\ln(s)]^2\,ds \\
&= \rho\left(\ln^2(\tau)-2\ln(\tau)\ln(\rho)+\ln^2(\rho)\right)+(\tau-\rho)\ln^2(\tau) \\
&- 2\ln\tau\int_\rho^\tau \ln(s)\,ds + \int_\rho^\tau \ln^2(s)\,ds \\
&= \rho\left(\ln^2(\tau)-2\ln(\tau)\ln(\rho)+\ln^2(\rho)\right)+(\tau-\rho)\ln^2(\tau)- \\
&- 2\ln(\tau)\left(\tau\ln(\tau)-\tau-\rho\ln(\rho)+\rho\right)+ \\
&+ \left(\tau\ln^2(\tau)-2\tau\ln(\tau)+2\tau\right)-\left(\rho\ln^2(\rho)-2\rho\ln(\rho)+2\rho\right) \\
&= \rho\ln^2(\tau)-2\rho\ln(\tau)\ln(\rho)+\rho\ln^2(\rho)+(\tau-\rho)\ln^2(\tau)- \\
&- 2\tau\ln^2(\tau)+2\tau\ln(\tau)+2\rho\ln(\tau)\ln(\rho)-2\rho\ln(\tau)+ \\
&+ \tau\ln^2(\tau)-2\tau\ln(\tau)+2\tau-\rho\ln^2(\rho)+2\rho\ln(\rho)-2\rho \\
&= 2\tau-2\rho-2\rho\ln(\tau)+2\rho\ln(\rho) = 2(\tau-\rho)\left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^\tau \Omega(s,\tau)^{1/2} S\Omega'(s,\tau)^{1/2}\,ds &= \begin{bmatrix} \int_0^\tau \sigma_f^2(s)\,dsS_{ff} & \int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,dsS_{fh} \\ \int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,dsS'_{fh} & \int_0^\tau \sigma_h^2(s,\tau)\,dsS_{hh} \end{bmatrix} \\
&= (\tau-\rho)\begin{bmatrix} S_{ff} & \left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)S_{fh} \\ \left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)S'_{fh} & 2\left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)S_{hh} \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\tau \tilde{\omega}(s,\tau)\,\tilde{\omega}(s,\tau)'\,ds \\
&= \mu^{-1}G^{-1}[I_\ell, FB]\begin{bmatrix} \int_0^\tau \sigma_f^2(s)\,dsS_{ff} & \int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,dsS_{fh} \\ \int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,dsS'_{fh} & \int_0^\tau \sigma_h^2(s,\tau)\,dsS_{hh} \end{bmatrix}\begin{bmatrix} I_\ell \\ B'F' \end{bmatrix}G^{-1} \\
&= G^{-1}\frac{(\tau-\rho)}{\mu}[I_\ell, FB]\begin{bmatrix} S_{ff} & \left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)S_{fh} \\ \left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)S'_{fh} & 2\left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)S_{hh} \end{bmatrix}\begin{bmatrix} I_\ell \\ B'F' \end{bmatrix}G^{-1} \\
&\frac{(\tau-\rho)}{\mu}G^{-1}\left(S_{ff}+\left(1-\rho\frac{\ln\left(\frac{\tau}{\rho}\right)}{\tau-\rho}\right)\left(S_{fh}B'F'+FBS'_{fh}\right)+2FBS_{hh}B'F'\left(1-\rho\frac{\ln\left(\frac{\tau}{\rho}\right)}{\tau-\rho}\right)\right)G^{-1};
\end{aligned}$$

(iii) Rolling Case:

Case (a): $\tau - \rho \geq \rho$

$$\begin{aligned}
&\int_0^\tau \sigma_h(s,\tau)\,\sigma_f(s)\,ds \\
&= \int_0^\tau 1(s \geq \rho)\left(\frac{1}{\rho}\left[s \cdot 1(s \leq \rho) + \rho \cdot 1(\rho < s \leq \tau - \rho) + (\tau - s)\cdot 1(s > \tau - \rho)\right]\right) ds \\
&= \int_\rho^\tau \frac{1}{\rho}\left[\rho \cdot 1(\rho \leq s < \tau - \rho) + (\tau - s)\cdot 1(s > \tau - \rho)\right] ds \\
&= \frac{1}{\rho}\left(\int_\rho^{\tau-\rho} \rho ds + \int_{\tau-\rho}^\tau (\tau - s)\,ds\right) = \frac{1}{\rho}\left(\rho(\tau - 2\rho) + \tau\rho - \frac{1}{2}\tau^2 + \frac{1}{2}(\tau-\rho)^2\right) \\
&= \tau - \frac{3}{2}\rho
\end{aligned}$$

$$\begin{aligned}
&\int_0^\tau \sigma_h^2(s,\tau)\,ds \\
&= \int_0^\tau \left(\frac{1}{\rho}\{s \cdot 1(s \leq \rho) + \rho \cdot 1(\rho < s \leq \tau - \rho) + (\tau - s)\cdot 1(s > \tau - \rho)\right)^2 ds \\
&= \left(\frac{1}{\rho}\right)^2\left(\int_0^\tau s^2 \cdot 1(s \leq \rho)\,ds + \int_0^\tau \rho^2 \cdot 1(\rho < s \leq \tau - \rho)\,ds + \int_0^\tau (\tau - s)^2 \cdot 1(s > \tau - \rho)\,ds\right) \\
&= \left(\frac{1}{\rho}\right)^2\left(\int_0^\rho s^2 ds + \rho^2 \int_\rho^{\tau-\rho} ds + \left(\frac{1}{\rho}\right)^2 \int_{\tau-\rho}^\tau (\tau - s)^2\,ds\right) \\
&= \left(\frac{1}{\rho}\right)^2 \frac{1}{3}\rho^3 + (\tau - 2\rho) + \left(\frac{1}{\rho}\right)^2\left(\tau^2\rho + \frac{\tau^3 - (\tau-\rho)^3}{3} - 2\tau\frac{\tau^2 - (\tau-\rho)^2}{2}\right) \\
&= \tau - \frac{4}{3}\rho
\end{aligned}$$

Thus,

$$\int_0^\tau \Omega(s,\tau)^{1/2} S\Omega'(s,\tau)^{1/2}\,ds = \begin{bmatrix} (\tau-\rho)S_{ff} & \left(\tau - \frac{3}{2}\rho\right)S_{fh} \\ \left(\tau - \frac{3}{2}\rho\right)S'_{fh} & \left(\tau - \frac{4}{3}\rho\right)S_{hh} \end{bmatrix},$$

and

$$\int_0^\tau \tilde{\omega}(s,\tau)\,\tilde{\omega}(s,\tau)'\,ds = \mu^{-1}G^{-1}\left[I_\ell, FB\right]\begin{bmatrix} (\tau-\rho)S_{ff} & \left(\tau - \frac{3}{2}\rho\right)S_{fh} \\ \left(\tau - \frac{3}{2}\rho\right)S'_{fh} & \left(\tau - \frac{4}{3}\rho\right)S_{hh} \end{bmatrix}\begin{bmatrix} I_\ell \\ B'F' \end{bmatrix} G^{-1} =$$

$$\mu^{-1}G^{-1}\left(S_{ff}(\tau-\rho) + \left(\tau - \frac{3}{2}\rho\right)\left(FBS'_{fh} + S_{fh}B'F'\right) + \left(\tau - \frac{4}{3}\rho\right)FBS_{hh}B'F'\right)G^{-1};$$

Case (b): $\tau - \rho < \rho$

$$\int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,ds$$
$$= \int_0^\tau \{1(s \geq \rho)\}\left\{\frac{s}{\rho}\cdot 1(s \leq \tau-\rho) + \frac{(\tau-\rho)}{\rho}\cdot 1(\tau-\rho < s \leq \rho) + \frac{(\tau-s)}{\rho}\cdot 1(s>\rho)\right\} ds =$$
$$= \int_0^\tau \frac{1}{\rho}(\tau - s)\cdot 1(s \geqslant \rho)\,ds = \frac{1}{\rho}\int_\rho^\tau (\tau-s)\,ds = \frac{1}{2\rho}(\tau-\rho)^2$$

$$\int_0^\tau \sigma_h^2(s)\,ds = \int_0^\tau \left(\frac{1}{\rho}\{s\cdot 1(s\leq\tau-\rho) + (\tau-\rho)\cdot 1(\tau-\rho<s\leq\rho) + (\tau-s)\cdot 1(s>\rho)\}\right)^2 ds$$
$$= \left(\frac{1}{\rho}\right)^2\left(\int_0^{\tau-\rho} s^2 ds + \int_{\tau-\rho}^{\rho}(\tau-\rho)^2 ds + \int_\rho^\tau (\tau-s)^2 ds\right)$$
$$= \left(\frac{1}{\rho}\right)^2\left(\frac{1}{3}(\tau-\rho)^3 + (\tau-\rho)^2(2\rho-\tau) + \tau^2(\tau-\rho) + \frac{\tau^3-\rho^3}{3} - \tau^3 + \tau\rho^2\right)$$
$$= \frac{1}{3\rho^2}(\tau-\rho)^2(4\rho-\tau)$$

Thus,

$$\int_0^\tau \Omega(s,\tau)^{1/2} S\Omega'(s,\tau)^{1/2}\,ds = \begin{bmatrix} (\tau-\rho)S_{ff} & \frac{1}{2\rho}(\tau-\rho)^2 S_{fh} \\ \frac{1}{2\rho}(\tau-\rho)^2 S_{fh}' & \frac{1}{3\rho^2}(\tau-\rho)^2(4\rho-\tau)S_{hh}\end{bmatrix},$$

and

$$\int_0^\tau \tilde{\omega}(s,\tau)\,\tilde{\omega}(s,\tau)'\,ds$$
$$= \mu^{-1}G^{-1}[I_\ell, FB]\begin{bmatrix} (\tau-\rho)S_{ff} & \frac{1}{2\rho}(\tau-\rho)^2 S_{fh} \\ \frac{1}{2\rho}(\tau-\rho)^2 S_{fh}' & \frac{(\tau-\rho)^2}{3\rho^2}(4\rho-\tau)S_{hh}\end{bmatrix}\begin{bmatrix} I_\ell \\ B'F'\end{bmatrix} G^{-1}$$
$$= \frac{(\tau-\rho)}{\mu}G^{-1}\left\{S_{ff} + \frac{(\tau-\rho)}{2\rho}\left(FBS_{fh}' + S_{fh}B'F'\right) + \frac{(\tau-\rho)(4\rho-\tau)}{3\rho^2}FBS_{hh}F'B'\right\}G^{-1}.$$

■

Proof of Theorem 5. The proof follows directly from Propositions 2 and 3. ■

Proof of Proposition 6. The result follows directly from Propositions 3 and 4 by imposing $F = 0$. ■

Proof of Proposition 7. From Proposition 2 note that:

$$\int_0^\tau \widetilde{\omega}(s,\tau)\,\widetilde{\omega}(s,\tau)'\,ds = \mu^{-1}G^{-1}\int_0^\tau [I_\ell, FB]\,\Omega(s,\tau)^{1/2} S\Omega(s,\tau)^{1/2}[I_\ell, FB]'\,dsG^{-1} \tag{20}$$
$$= \mu^{-1}G^{-1}\int_0^\tau [\sigma_f^2(s)\,S_{ff} + \sigma_f(s)\,\sigma_h(s,\tau)\,(FBS_{hf} + S_{fh}B'F') + \sigma_h^2(s,\tau)\,FBS_{hh}B'F']dsG^{-1}$$

(i) Recursive case:

By imposing $-\frac{1}{2}(FBS_{hf}+S_{fh}B'F')=FBS_{hh}B'F'$ from condition (10) on equation (20), we can have:

$$\int_0^\tau \tilde{\omega}(s,\tau)\,\tilde{\omega}(s,\tau)'\,ds = \mu^{-1}G^{-1}\int_0^\tau \begin{bmatrix} \sigma_f^2(s)\,S_{ff}+\sigma_f(s)\,\sigma_h(s,\tau)\,(FBS_{hf}+S_{fh}B'F') \\ +\sigma_h^2(s,\tau)\,FBS_{hh}B'F' \end{bmatrix} dsG^{-1}$$
$$= \mu^{-1}G^{-1}\int_0^\tau \begin{bmatrix} \sigma_f^2(s)\,S_{ff}-2\sigma_f(s)\,\sigma_h(s,\tau)\,FBS_{hh}B'F' \\ +\sigma_h^2(s,\tau)\,FBS_{hh}B'F' \end{bmatrix} dsG^{-1}.$$

From Proposition 4 note that $\int_0^\tau \sigma_h^2(s,\tau)\,ds = 2\ \int_0^\tau \sigma_h(s,\tau)\,\sigma_f(s)\,ds$. This further simplifies the expression:

$$\int_0^\tau \tilde{\omega}(s,\tau)\,\tilde{\omega}(s,\tau)'\,ds = \mu^{-1}G^{-1}\int_0^\tau \begin{bmatrix} \sigma_f^2(s)\,S_{ff}-2\sigma_f(s)\,\sigma_h(s,\tau)\,FBS_{hh}B'F' \\ +\sigma_h^2(s,\tau)\,FBS_{hh}B'F' \end{bmatrix} dsG^{-1}$$
$$= \mu^{-1}G^{-1}\left(\int_0^\tau \sigma_f^2(s)\,S_{ff}ds\right)G^{-1} = \frac{(\tau-\rho)}{\mu}G^{-1}S_{ff}G^{-1}$$

(ii) Rolling case:

By imposing condition (10) on equation (20), we can further simplify:

$$\int_0^\tau \widetilde{\omega}(s,\tau)\,\widetilde{\omega}(s,\tau)'\,ds = \mu^{-1}G^{-1}S_{ff}G^{-1}\int_0^\tau [\sigma_f(s)^2 - 2\sigma_f(s)\,\sigma_h(s,\tau)+\sigma_h^2(s,\tau)]ds. \quad (21)$$

Case (a): $\tau-\rho \geqslant \rho$.

By Proposition 4, equation (21) simplifies to:

$$\int_0^\tau \tilde{\omega}(s,\tau)\,\tilde{\omega}(s,\tau)'\,ds = \mu^{-1}\int_0^\tau [\sigma_f(s)^2 - 2\sigma_f(s)\,\sigma_h(s,\tau)+\sigma_h^2(s,\tau)^2]dsG^{-1}S_{ff}G^{-1} \quad (22)$$
$$= \frac{2}{3}\frac{\rho}{\mu}G^{-1}S_{ff}G^{-1},$$

and is independent of τ.

Case (b): $\tau-\rho<\rho$.

By Proposition 4, equation (21) simplifies to:

$$\int_0^\tau \tilde{\omega}(s)\,\tilde{\omega}(s)'\,ds = \frac{1}{\mu}\int_0^\tau [\sigma_f^2(s) - 2\sigma_f(s)\,\sigma_h(s,\tau)+\sigma_h^2(s,\tau)]dsG^{-1}S_{ff}G^{-1}$$
$$= \frac{1}{\mu}\left[(\tau-\rho) - 2\frac{1}{2\rho}(\tau-\rho)^2 + \frac{1}{3\rho^2}(\rho-\tau)^2(4\rho-\tau)\right]G^{-1}S_{ff}G^{-1}$$
$$= \left(\frac{\tau-\rho}{\mu}\right)\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)G^{-1}S_{ff}G^{-1}. \quad (23)$$

Furthermore, from Proposition 2 we have that:

$$\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds = \mu^{-1}G^{-1}\int_0^\tau [I_\ell, FB]\left\{\begin{array}{c}\left[\Omega(s,\tau)^{1/2}-\Omega(s,\tau-\mu)^{1/2}\right]\cdot 1(s\le\tau-\mu)\\ +\Omega(s,\tau)^{1/2}\cdot 1(\tau-\mu<s\le\tau)\end{array}\right\}$$

$$\times S\left\{\begin{array}{c}\left[\Omega(s,\tau)^{1/2}-\Omega(s,\tau-\mu)^{1/2}\right]\cdot 1(s\le\tau-\mu)\\ +\Omega(s,\tau)^{1/2}\cdot 1(\tau-\mu<s\le\tau)\end{array}\right\}[I_\ell, FB]'\,dsG^{-1}$$

$$= \mu^{-1}G^{-1}\left\{\begin{array}{c}\left(\int_{\tau-\mu}^{\tau}\sigma_f^2(s)\,ds\right)S_{ff}+\\ \left(\int_{\tau-\mu}^{\tau}\sigma_f(s)\,\sigma_h(s,\tau)\,ds\right)[FBS_{hf}+S_{fh}B'F']+\\ \int_0^\tau\left[\begin{array}{c}(\sigma_h(s,\tau)-\sigma_h(s,\tau-\mu))^2\cdot 1(s\le\tau-\mu)\\ +\sigma_h^2(s,\tau)\cdot 1(\tau-\mu<s\le\tau)\end{array}\right]dsFBS_{hh}B'F'\end{array}\right\}G^{-1} \quad (24)$$

(i') Recursive case:

Similar to (i), by imposing $-\frac{1}{2}(FBS_{hf}+S_{fh}B'F') = FBS_{hh}B'F'$ from condition (10) and from Proposition 3 the fact that

$$\int_0^\tau\left[\begin{array}{c}(\sigma_h(s,\tau)-\sigma_h(s,\tau-\mu))^2\cdot 1(s\le\tau-\mu)\\ +\sigma_h^2(s,\tau)\cdot 1(\tau-\mu<s\le\tau)\end{array}\right]ds = 2\int_{\tau-\mu}^{\tau}\sigma_f(s)\,\sigma_h(s,\tau)\,ds,$$

we can simplify equation (24) to:

$$\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds = \mu^{-1}G^{-1}\left(\int_{\tau-\mu}^{\tau}\sigma_f^2(s)\,ds\right)G^{-1} = \frac{\mu}{\mu}G^{-1}S_{ff}G^{-1} = G^{-1}S_{ff}G^{-1}.$$

(ii') Rolling case: Let $\pi^\dagger \equiv \frac{\mu}{\rho}$.

Further, under condition (10), we have:

$$\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds = \mu^{-1}G^{-1}S_{ff}G^{-1}\times \quad (25)$$

$$\left[\int_{\tau-\mu}^{\tau}\sigma_f^2(s)\,ds - 2\left(\int_{\tau-\mu}^{\tau}\sigma_h(s,\tau)\,\sigma_f(s)\,ds\right)+\int_0^\tau\left(\begin{array}{c}[\sigma_h(s,\tau)-\sigma_h(s,\tau-\mu)]^2\cdot 1(s\le\tau-\mu)\\ +\sigma_h^2(s,\tau)\cdot 1(\tau-\mu<s\le\tau)\end{array}\right)ds\right].$$

By Proposition 3, equation (25) simplifies to:

Case (a): If $\mu \geqslant \rho$, then $\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds = \frac{2}{3\pi^\dagger}$;

Case (b): If $\mu < \rho$, then $\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds = 1-\frac{1}{3}\left(\pi^\dagger\right)^2$; ∎

Proof of Theorem 8. (a) From Proposition 2 and Theorem 5 we have

$$\begin{aligned}
\mathcal{W}_{t,m} &= m\widehat{\theta}_j V_{\theta,\tau}^{-1}\widehat{\theta}_j \\
&\Rightarrow \left[\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)' ds\right) - \mathcal{B}_\ell\left(\int_0^{\tau-\mu} \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)' ds\right)\right]' V_{\theta,\tau}^{-1}\times \\
&\quad \times \left[\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)' ds\right) - \mathcal{B}_\ell\left(\int_0^{\tau-\mu} \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)' ds\right)\right]
\end{aligned}$$

Under condition (10) and by the results in Proposition 7:

(i) Recursive case:

$V_{\theta,\tau} = V_\theta = G^{-1}S_{ff}G^{-1}$ and $\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)' ds\right) = \left(G^{-1}S_{ff}G^{-1}\right)^{1/2}\mu^{-1/2}\mathcal{B}_\ell(\tau-\rho)$.

Thus,

$\mathcal{W}_{t,m} \Rightarrow \mu^{-1}\left[\mathcal{B}_\ell(\tau-\rho) - \mathcal{B}_\ell(\tau-\mu-\rho)\right]'\left[\mathcal{B}_\ell(\tau-\rho) - \mathcal{B}_\ell(\tau-\mu-\rho)\right]$.

(ii) Rolling case:

From Proposition 7, $V_{\theta,\tau} = V_\theta = \left(G^{-1}S_{ff}G^{-1}\right)\left[\left(\frac{2}{3\pi^\dagger}\right)\cdot 1(\mu \geq \rho) + \left(1 - \frac{1}{3}\left(\pi^\dagger\right)^2\right)\cdot 1(\mu < \rho)\right]$.

Furthermore, when $\tau - \rho < \rho$,

$$\begin{aligned}
\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)' ds\right) &= \mathcal{B}_\ell\left(\left(\frac{\tau-\rho}{\mu}\right)\left(1 - \frac{(\tau-\rho)^2}{3\rho^2}\right)G^{-1}S_{ff}G^{-1}\right) \\
&= \left(G^{-1}S_{ff}G^{-1}\right)^{1/2}\mu^{-1/2}\mathcal{B}_\ell\left((\tau-\rho)\left(1 - \frac{(\tau-\rho)^2}{3\rho^2}\right)\right).
\end{aligned}$$

Alternatively, when $\tau - \rho \geqslant \rho$,

$$\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)' ds\right) = \left(G^{-1}S_{ff}G^{-1}\right)^{1/2}\mu^{-1/2}\mathcal{B}_\ell\left(\frac{2}{3}\rho\right).$$

Thus,

$$\begin{aligned}
&\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)' ds\right) \\
&= \left(G^{-1}S_{ff}G^{-1}\right)^{1/2}\mu^{-1/2}\left[\begin{array}{c} \mathcal{B}_\ell\left((\tau-\rho)\left(1 - \frac{(\tau-\rho)^2}{3\rho^2}\right)\right)\cdot 1(\mu+\rho \leq \tau < 2\rho) \\ +\mathcal{B}_\ell\left(\frac{2}{3}\rho\right)\cdot 1(\tau \geqslant 2\rho) \end{array}\right],
\end{aligned}$$

$$\begin{aligned}
&\mathcal{B}_\ell\left(\int_0^{\tau-\mu} \widetilde{\omega}(s,\tau-\mu)\widetilde{\omega}(s,\tau-\mu)' ds\right) \\
&= \left(G^{-1}S_{ff}G^{-1}\right)^{1/2}\mu^{-1/2}\left[\begin{array}{c} \mathcal{B}_\ell\left((\tau-\mu-\rho)\left(1 - \frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\cdot 1(\rho \leq \tau-\mu < 2\rho) \\ +\mathcal{B}_\ell\left(\frac{2}{3}\rho\right)\cdot 1(\tau-\mu \geqslant 2\rho) \end{array}\right].
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\,\widetilde{\omega}(s,\tau)'\,ds\right)-\mathcal{B}_\ell\left(\int_0^{\tau-\mu} \widetilde{\omega}(s,\tau-\mu)\,\widetilde{\omega}(s,\tau-\mu)'\,ds\right)\\
&=\left(G^{-1}S_{ff}G^{-1}\right)^{1/2}\mu^{-1/2}\left[\left\{\begin{array}{c}\mathcal{B}_\ell\left((\tau-\rho)\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)\right)\\ -\mathcal{B}_\ell\left((\tau-\mu-\rho)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\end{array}\right\}\cdot 1(\mu+\rho\leq\tau<2\rho)\right.\\
&\quad\left.\begin{array}{c}+\left[\mathcal{B}_\ell\left(\frac{2}{3}\rho\right)-\mathcal{B}_\ell\left((\tau-\mu-\rho)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\right]\cdot 1(2\rho\leq\tau<2\rho+\mu)\\ +0\cdot 1(\tau\geqslant 2\rho+\mu)\end{array}\right]\\
&\equiv\mathcal{B}_\ell\left(\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathcal{W}_{t,m}\Rightarrow\mathcal{B}_\ell\left(\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds\right)'V_{\theta,\tau}^{-1}\mathcal{B}_\ell\left(\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds\right)=\\
&\quad=\mu^{-1}\left\{\left(\frac{2}{3\pi^\dagger}\right)\cdot 1(\mu\geq\rho)+\left(1-\frac{1}{3}\left(\pi^\dagger\right)^2\right)\cdot 1(\mu<\rho)\right\}^{-1}\times\\
&\quad\left[\left\{\mathcal{B}_\ell\left((\tau-\rho)\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)\right)-\mathcal{B}_\ell\left((\tau-\mu-\rho)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\right\}\cdot 1(\mu+\rho\leq\tau<2\rho)\right.\\
&\quad\left.+\left\{\mathcal{B}_\ell\left(\frac{2}{3}\rho\right)-\mathcal{B}_\ell\left((\tau-\mu-\rho)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\right\}\cdot 1(2\rho<\tau\leq 2\rho+\mu)\right]'\\
&\quad\times\left[\left\{\mathcal{B}_\ell\left((\tau-\rho)\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)\right)-\mathcal{B}_\ell\left((\tau-\mu-\rho)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\right\}\cdot 1(\mu+\rho\leq\tau<2\rho)\right.\\
&\quad\left.+\left\{\mathcal{B}_\ell\left(\frac{2}{3}\rho\right)-\mathcal{B}_\ell\left((\tau-\mu-\rho)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\right\}\cdot 1(2\rho<\tau\leq 2\rho+\mu)\right].
\end{aligned}
$$

(b) Follows directly from Proposition 6 using the same arguments in the proof of (a). ■

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3 Additional Critical Value Tables

Table A.1a. Critical Values for the Fluctuation Rationality Test

Recursive Case

Panel A. 10% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	5.9556	5.4013	6.9650	6.4921
2	9.2245	8.5250	10.1145	9.7747
3	11.3230	10.2160	12.2866	11.7772
4	13.2317	12.1300	14.1436	13.4497
5	14.6695	13.6400	15.8784	15.3008
6	16.5875	15.5707	18.3062	17.8984
7	18.8531	17.6269	19.7706	19.4035
8	19.5394	18.2724	21.4959	20.6359
9	22.0456	20.9694	23.3407	22.6708
10	23.0327	21.7260	24.6531	23.7097

Panel B. 5% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	7.6103	6.8123	8.1529	8.0414
2	10.7828	10.3909	12.1409	11.1946
3	12.6497	11.8263	14.2097	13.1495
4	14.8763	14.2381	15.8727	15.4504
5	16.4838	16.1415	17.9421	17.4355
6	19.1056	18.1881	20.6415	19.9306
7	20.3082	19.5852	21.5059	20.9714
8	21.9336	20.6173	23.5220	22.9359
9	24.6251	23.6866	25.6793	25.6762
10	25.3684	24.1207	26.8021	25.7949

Panel C. 1% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	10.2989	9.6853	11.7167	10.7821
2	14.7120	14.8595	16.8443	16.0830
3	17.1015	16.0334	19.4287	16.9283
4	17.8342	17.2788	18.7771	18.4869
5	21.2392	20.5449	22.3047	23.3458
6	23.1487	22.8962	23.6097	23.9932
7	24.5539	23.6424	25.8011	26.0693
8	27.4389	25.7965	28.1082	27.6579
9	29.1371	27.9073	30.7444	28.9690
10	29.8898	28.5949	32.6835	29.7556

Note. The table reports the critical values, $\kappa_{\alpha,\ell}$, for several restrictions (ℓ) at α =10%, 5% and 1% significance levels for $max_{j\in\{R+m,\ldots,T\}}\ \mathcal{W}_{j,m}$ for the recursive scheme under condition (10). Critical values are based on $T = 1000$ and 1000 Monte Carlo simulations; $\rho = R/T$ and $\mu = m/T$.

Table A.1b. Critical Values for the Fluctuation Rationality Test

Rolling Case

Panel A. 10% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	6.4740	5.5899	7.7008	7.2863
2	9.1746	8.5913	10.7956	10.6739
3	11.5663	10.8638	12.9363	12.3039
4	13.2258	12.6272	14.4234	13.9984
5	14.9127	14.6108	17.0396	16.5904
6	16.4976	15.6346	19.0793	18.3600
7	18.7578	17.7502	20.5095	19.8632
8	19.9329	19.2351	22.7822	22.0157
9	21.7496	21.2777	23.9299	23.9710
10	23.1398	22.2947	25.2720	24.8407

Panel B. 5% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	7.7122	6.9621	8.8102	8.5989
2	10.5702	10.0698	12.4778	12.1265
3	13.2956	12.3069	14.5513	13.9501
4	14.8771	14.2805	16.6307	15.6392
5	16.8451	16.6441	19.0969	18.6127
6	18.5144	17.5945	20.9080	20.1921
7	21.0426	19.7563	22.6405	21.9120
8	22.9293	21.4715	25.1263	24.2192
9	24.2818	23.4890	26.4981	26.1489
10	25.7621	24.6734	26.8059	26.7959

Panel C. 1% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	11.0943	10.3372	11.7440	11.0335
2	13.7842	14.1051	15.6558	15.7895
3	17.2460	16.4848	18.3441	18.0914
4	18.3709	18.0079	21.0888	21.4351
5	21.6826	20.6014	22.3863	22.5591
6	22.7820	22.9287	25.9931	25.1753
7	24.9451	23.4760	26.5961	25.3780
8	27.9311	26.4139	28.7270	28.1257
9	30.1262	28.8866	32.1298	30.3945
10	31.0603	30.4121	31.7719	32.4430

Note. The table reports the critical values, $\kappa_{\alpha,\ell}$, for several restrictions (ℓ) at α =10%, 5% and 1% significance levels for $max_{j\in\{R+m,\ldots,T\}}\ \mathcal{W}_{j,m}$ for the rolling scheme under condition (10). Critical values are based on $T = 1000$ and 1000 Monte Carlo simulations; $\rho = R/T$ and $\mu = m/T$.

Table A.1c. Critical Values for the Fluctuation Rationality Test

Survey and Model-Free Forecasts

Panel A. 10% Significance Level

	$\widetilde{\mu}$								
ℓ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	10.0909	8.8274	7.7116	6.9555	6.4272	5.8410	4.9404	4.8508	4.0096
2	13.2456	11.4773	10.7955	9.6482	9.3648	8.3442	7.7478	7.4669	6.2243
3	15.9915	14.2049	13.3396	11.6461	11.4939	10.4839	9.3900	8.9699	7.9423
4	18.4447	15.5897	15.1254	13.8661	13.2415	12.7312	11.5331	10.7335	9.3509
5	19.9690	18.2447	16.7190	15.7116	15.0672	14.1355	13.1798	12.1317	11.4857
6	22.3413	20.3183	19.1924	18.0867	17.3395	15.6658	14.7640	13.4408	12.3823
7	24.3462	22.3182	21.0891	20.0822	18.4705	17.7375	16.8276	15.6473	13.9220
8	26.5930	23.8825	22.4070	21.6763	20.1078	18.7631	18.1456	17.0475	15.8832
9	27.8409	26.0117	24.5040	23.3912	21.4939	20.6265	19.2173	18.0560	17.2398
10	29.2933	27.1103	25.8483	24.6502	23.4647	22.5659	20.6670	20.0081	18.4020

Panel B. 5% Significance Level

	$\widetilde{\mu}$								
ℓ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	11.8290	10.5637	8.9252	8.1468	8.1409	7.2803	6.4978	6.0837	5.4695
2	14.9966	13.0846	12.8141	10.9084	11.1314	9.9386	9.1724	9.0589	7.8305
3	17.6768	15.7548	15.0608	13.4383	13.2113	12.6018	10.9597	10.8426	9.4727
4	19.8434	17.6051	17.0158	16.3186	15.1404	14.7573	13.5928	13.1087	10.8243
5	21.7091	20.4659	18.7186	18.2152	17.1092	15.6317	15.4842	13.9418	13.6335
6	24.2721	22.4870	20.9717	20.2839	20.2971	17.8602	16.5583	15.4633	14.4789
7	26.2869	24.2644	22.8543	21.6818	20.5974	20.1200	19.0697	17.7064	15.9126
8	28.3030	25.7461	24.3315	23.4497	22.4328	21.1563	20.3632	19.1440	18.1475
9	29.5489	27.9249	26.8101	25.2662	24.2510	22.7821	21.7109	20.2745	19.7147
10	31.7548	29.4709	27.5980	27.0357	25.3011	25.3250	23.4556	22.6180	21.6647

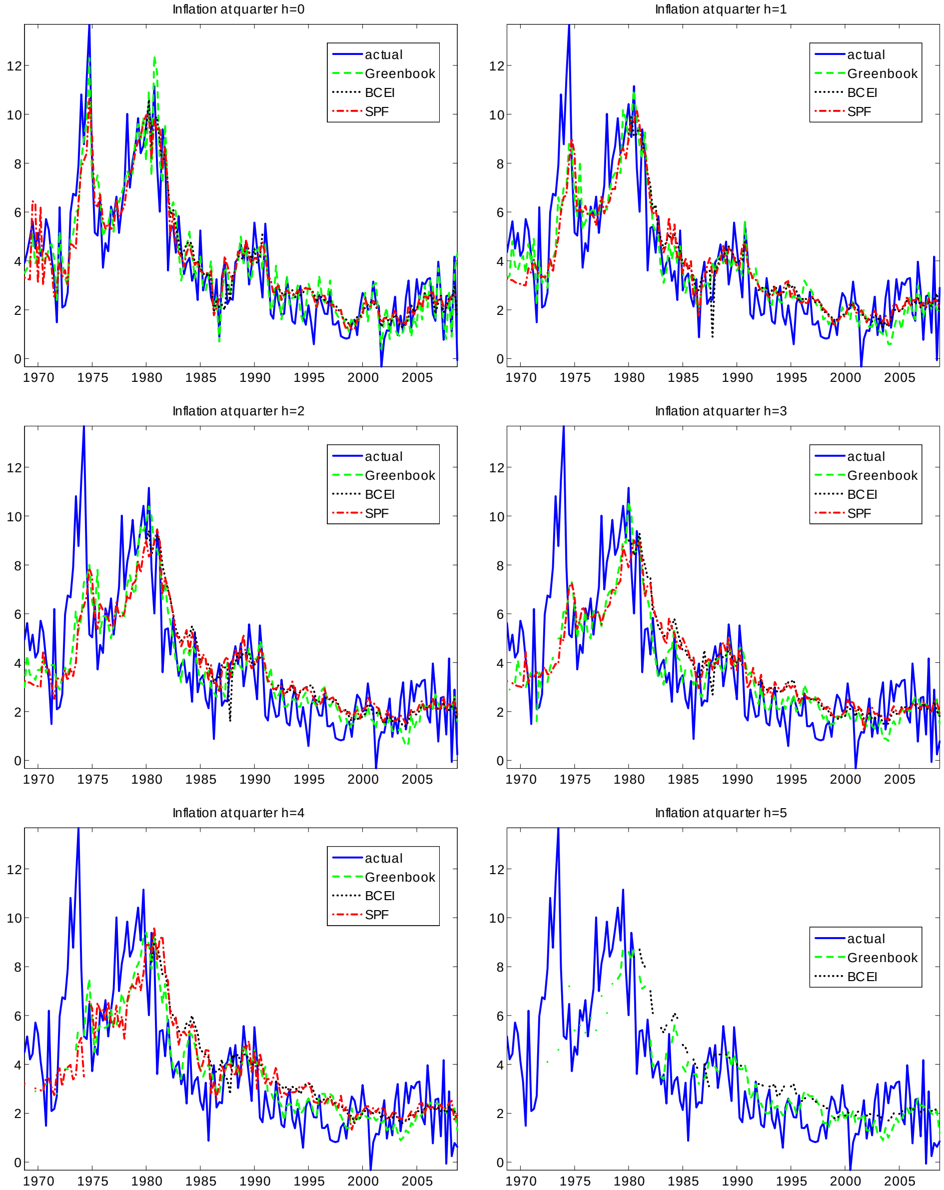
Panel B. 1% Significance Level

	$\widetilde{\mu}$								
ℓ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	15.0484	14.0966	12.0133	12.5257	11.2177	10.5051	9.4237	8.5435	8.2030
2	18.5865	15.9096	16.8898	14.1998	14.0067	14.1862	12.5407	12.8689	10.4675
3	22.2589	18.4610	18.1309	16.6749	16.6152	16.6209	15.1820	14.2981	12.7461
4	23.2931	22.6241	20.4215	20.1829	18.5263	18.7161	17.1985	16.8407	15.3503
5	25.9859	24.9017	22.1130	22.3233	22.0671	19.2611	19.5215	17.5694	18.9347
6	28.3701	27.1948	25.1033	24.2014	24.8307	22.2983	20.9119	19.1080	18.0740
7	31.4425	28.9881	27.1075	26.9725	24.3925	24.0897	24.2268	21.7648	20.5248
8	32.4669	29.5149	28.6144	28.1559	26.8484	26.2502	25.2785	23.0942	22.8633
9	33.7114	35.2864	31.3305	29.5492	29.0904	28.2170	25.6922	26.0238	24.8479
10	36.6704	33.1601	32.2650	32.6074	30.0138	31.1782	27.3047	25.9979	26.2313

Note. The table reports the critical values, $\kappa_{\alpha,\ell}$, for several restrictions (ℓ) at α =10%, 5% and 1% significance levels for $max_{j\in\{R+m,...,T\}}\ \mathcal{W}_{j,m}$ for the case when parameter estimation error is irrelevant as in Corollary 9 in the paper. Critical values are based on $P = 1000$ and 1000 Monte Carlo simulations; $\tilde{\mu} = m/P$.

4 Additional Data Figures

Figure A.1: Inflation Forecasts



Note. The figure plots Greenbook, BCEI and SPF forecasts of inflation for various forecast horizons h in conjunction with the realized values of inflation, labeled as "actual," for the corresponding horizon. If a forecast for a specific horizon by the corresponding agency does not exist, it is depicted as a missing value.